18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #11 **Due: 24 Nov 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Let $X = \mathbb{P}^n$ and $p = [1 : 0 : \cdots : 0]$, and let $\mathcal{M}$ be the skyscraper sheaf supported at p, i.e.,

$$\mathcal{M}(U) = \begin{cases} \mathbb{C} & \text{if } p \in U, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Compute the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{M})$ with respect to the standard affine open cover $\mathcal{U}$.

   (b) Compute the Čech cohomology groups $\check{H}^q(\mathcal{U}, \mathcal{M})$ for all $q \geq 0$.

2. Let $d \geq 0$ and consider the twisting module $\mathcal{O}(d)$ on $X := \mathbb{P}^1$.

   (a) Compute the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{O}(d))$ with respect to the standard affine open cover $\mathcal{U}$.

   (b) Compute the Čech cohomology groups $\check{H}^q(\mathcal{U}, \mathcal{O}(d))$ for all $q \geq 0$. Verify that your answer agrees with the results on the cohomology groups of twisting modules proved in class via the long exact sequence.

3. Let X be a variety. Recall that an invertible sheaf $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank 1, so there exists an affine open cover $\mathcal{U} = \{U_i\}_i$ of X and isomorphisms $\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$ for each i. Then

$$\varphi_i(U_{ij}) \circ \varphi_j(U_{ij})^{-1} : \mathcal{O}_X(U_{ij}) \xrightarrow{\sim} \mathcal{O}_X(U_{ij})$$

   is an isomorphism of $\mathcal{O}_X(U_{ij})$-modules for each i, j, where $U_{ij} = U_i \cap U_j$. Thus $\varphi_i(U_{ij}) \circ \varphi_j(U_{ij})^{-1}$ is given by multiplication by an invertible function $g_{ij} \in \mathcal{O}_X^\times(U_{ij})$, namely

$$g_{ij} = (\varphi_i(U_{ij}) \circ \varphi_j(U_{ij})^{-1})(1)\,,$$

   where $\mathcal{O}_X^\times$ is the sheaf of invertible regular functions.

   (a) Show that $g := (g_{ij})_{ij} \in \check{Z}^1(\mathcal{U}, \mathcal{O}_X^\times)$ is a Čech 1-cocycle. (Note that $\mathcal{O}_X^\times(V)$ is a group under *multiplication* for each open V.)

(b) Suppose we have another local trivialization of $\mathcal{L}$ over $\mathcal{U}$, i.e., another family $\{\psi_i\}_i$ of isomorphisms $\psi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$. Let $h = (h_{ij})_{ij}$ be the corresponding 1-cocycle, as constructed in the previous part. Show that g and h differ (multiplicatively) by a 1-coboundary, i.e., an element of $\check{B}^1(\mathcal{U}, \mathcal{O}_X^\times)$. (With a little more work, one can show that $\operatorname{Pic}(X) \cong \check{H}^1(\mathcal{U}, \mathcal{O}_X^\times)$.)

4. Let A, B be 2×2 variable matrices, let $P = \mathbb{C}[a_{ij}, b_{ij}]$ be the polynomial ring in their entries, and let R be the quotient of P by the ideal that expresses the condition that $AB = BA$. Show that R has a resolution as a P-module of the form

$$0 \to P^2 \to P^3 \to P \to R \to 0\,,$$

i.e., one can construct an exact sequence of the form above. (*Hint*: Write the equations in terms of $a_{11} - a_{22}$ and $b_{11} - b_{22}$.)