18.721 Introduction to Algebraic Geometry
Fall 2021

Problem Set #10 **Due: 19 Nov 2021**

Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Let $0 \to V_0 \to \cdots \to V_n \to 0$ be a complex of finite-dimensional vector spaces. Show that
$$\sum_i (-1)^i \dim(V_i) = \sum_q (-1)^q \dim(H^q(V^\bullet))\,.$$

2. On the projective line $\mathbb{P}^1$, when are two direct sums $\mathcal{O}(m) \oplus \mathcal{O}(n)$ and $\mathcal{O}(r) \oplus \mathcal{O}(s)$ isomorphic? (*Hint*: Use that twisting is an exact functor.)

3. (a) Let $X = \mathbb{P}^n$ and K be the function field of X. Let $\mathcal{F}$ be the constant function field $\mathcal{O}_X$-module, that is $\mathcal{F}(U) = K$ for any nonempty open $U \subseteq X$. Prove that $H^q(X, \mathcal{F}) = 0$ for all $q > 0$.

   (b) Let $X = \mathbb{P}^n$ and $\{U^i\}_i$ be an open covering of X. Suppose that rational functions f_i are given such that $f_i - f_j$ is a regular function on $U^{ij} := U^i \cap U^j$ for all i, j. Let $\mathcal{Q} = \mathcal{F}/\mathcal{O}$ and consider the exact sequence
$$0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{Q} \to 0\,.$$
   Using this exact sequence, solve the *Cousin Problem*: is there a rational function f such that $f - f_i$ is a regular function on V^i for each i?

4. (a) Let $X = \mathbb{P}^2 \setminus \{(0:0:1)\}$. Use the covering of X by the two standard affine open sets $\mathbb{U}^0, \mathbb{U}^1$ to compute the cohomology $H^q(X, \mathcal{O}_X)$.

   (b) Suppose Y is a variety covered by two affine open sets, and $\mathcal{M}$ is an $\mathcal{O}_Y$-module. Show that $H^q(Y, \mathcal{M}) = 0$ for all $q > 1$.