**18.721 Introduction to Algebraic Geometry**
**Fall 2021**

**Problem Set #1** **Due: 09/17/2021**

---

## Description

Solutions are to be prepared in typeset form (typically via latex) and submitted electronically as a pdf file via Gradescope by midnight on the due date. Late problem sets will not be accepted – your lowest score is dropped, so you can afford to skip one problem set without penalty.

Collaboration is permitted/encouraged, but you must write up your own solutions and explicitly identify your collaborators, as well as any resources you consulted that are not listed above. If there are none, please indicate this by writing **Sources consulted: none** at the top of your submission.

1. Let $f(x, y, z)$ be an homogeneous irreducible polynomial of degree $> 1$. Prove that the locus $f = 0$ in $\mathbb{P}^2$ contains three points that do not lie on a line.

2. (a) Consider the curve $X\colon y^2 = x^3 \subset \mathbb{A}^2$. Construct a surjective map $\varphi : \mathbb{A}^1 \to X \subset \mathbb{A}^2$, i.e., find a parametrization of $X$.

   (b) Consider the induced map on the algebras

   $$\begin{aligned} \varphi^*\colon \mathbb{C}[x, y]/(y^2 - x^3) &\to \mathbb{C}[t] \\ g(x, y) &\mapsto g(\varphi(t)), \end{aligned}$$

   Show that $\varphi^*$ is not an isomorphism.

   (c) In class, we saw that for $Y\colon y^2 = x^3 + x^2$ the map $\varphi(t) := (t^2 - 1, t^3 - t)$ is a parametrization for $Y$. Show that in this setting,

   $$\begin{aligned} \varphi^*\colon \mathbb{C}[x, y]/(y^2 - x^3 - x^2) &\to \mathbb{C}[t] \\ g(x, y) &\mapsto g(t^2 - 1, t^3 - t), \end{aligned}$$

   is also not an isomorphism.

3. Show that a cubic curve in $\mathbb{P}^2$, i.e., defined by a degree 3 homogeneous irreducible polynomial, has at most 1 singular point.

4. Let $C$ be a smooth cubic curve in $\mathbb{P}^2$ defined by $f$, and let $(0, 1, 0)$ be a flex point of $C$ whose tangent is $z = 0$.

   (a) Show that the coefficients of $x^2y$, $xy^2$, and $y^3$ in $f$ are zero.

   (b) Show that with a suitable choice of coordinates, one can reduce the defining polynomial to the form $f = y^2z + x^3 + axz^2 + bz^3$, where $x^3 + ax + b$ is a polynomial with distinct roots.

   (c) Show that after a change of variables, one can take either $a = 1$ or $b = 1$.

5. Let $f(x)$ be a polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. Show that $\{x : f(x) = 0\} = \mathbb{A}^n$ if and only if $f$ is the zero polynomial.

6. Let $p$ be a prime number. Recall that $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is a field.

(a) Consider the polynomial $g(x, y) = x^2y + y^2x \in \mathbb{F}_2[x, y]$. Show that $g(x, y) = 0$ for every $(x, y) \in \mathbb{F}_2^2$.

(b) Find a nonzero polynomial in $\mathbb{F}_2[x, y, z]$ involving all three variables that vanishes at every point of $\mathbb{F}_2^3$.

(c) Let $p$ be any prime number. Find a nonzero polynomial in $\mathbb{F}_p[x]$ that vanishes at all points of $\mathbb{F}_p$. (Hint: Fermat's little theorem )