18.705 Commutative Algebra **Fall 2026**

Problem Set #6 **Due: 11/25/2026**

These problems are related to material covered in Lectures 16–18.

Instructions: The format of this problem set is the same as that of Problem Sets 4 and 5. Problem 1 consists of eight exercises (a)–(h) from AK13 whose statements are reproduced below; solve all eight. Write up complete solutions to *at least two* of them, and clearly indicate which two you want graded (you are welcome to write up more than two, but only the two you indicate will be graded). For each exercise you do not write up, check your answer against the solution in AK13 (clicking the exercise number in AK13 will take you directly to the solution, be careful not to click prematurely) and, in place of a write-up, simply write the sentence "I checked my answer against the solution in AK13." Then complete the survey problem 2. Your answer should be in the form of a PDF submitted to Gradescope no later than midnight on the date due (do not ask for an extension, the answer is no). Be sure to include your name somewhere on the PDF. Each of the two graded exercises is worth 1 point and the grading is binary; any credible attempt will be awarded 1 point.

I recommend using latex to create the PDF, but scanned handwritten solutions are fine provided your handwriting is legible. For Problem 1 collaboration (including with LLMs) is permitted, but I strongly encourage you to first attempt the problems on your own. Problem 2 should be answered entirely on your own.

Important: Be sure to use the 2013 version of Altman–Kleiman; newer versions do not necessarily use the same numbering.

Problem 1. Problems from Altman–Kleiman (2 points)

Read Chapter 22 of AK13, together with (20.11), (20.12), and (20.16)–(20.19) of Chapter 20 (filtrations, Rees algebras, and the Artin–Rees lemma), (23.15)–(23.24) of Chapter 23 (Serre's conditions), and (24.1)–(24.13) of Chapter 24 (Dedekind domains); Lectures 16–18 cover much of this material. The total ring of fractions and normal rings that are not domains, treated at the start of Lecture 16, appear in (14.13)–(14.17), which were part of the reading for Problem Set 4. You may continue to skip the rest of Chapter 20, and you may skip the rest of Chapter 24. Three differences between AK13 and the lectures are worth keeping in mind as you read:

- AK13 defines Serre's condition (S_n) in terms of depth, which we have not yet covered; see (23.4) and (23.5). For a Noetherian domain R and a nonzero prime $\mathfrak{p}$ of R, it follows from Lemma (23.5) that $\operatorname{depth}(R_{\mathfrak{p}}) = 1$ if and only if $\mathfrak{p}$ is associated to a nonzero principal ideal of R. With this translation, (S_2) for R is the condition that every prime associated to a nonzero principal ideal has height one, and (23.18)–(23.20) are Serre's criterion as stated in Lecture 16.
- In AK13 a module is *complete* if every Cauchy sequence has a limit, and limits need not be unique. The I-adically complete modules of Lecture 17 (those for which $M \to \widehat{M}$ is an isomorphism) are the modules that AK13 calls separated and complete.

- In AK13 a Dedekind domain is a Noetherian normal domain of dimension exactly one, while the definition in Lecture 16 (dimension at most one) also allows fields. Exercises (b)–(d) below hold with either definition.

(a) (Exercise 23.17) Let R be a Noetherian ring. Prove that R is reduced if and only if it satisfies both of the following conditions:

(R_0) $R_{\mathfrak{p}}$ is a field for every minimal prime $\mathfrak{p}$ of R;

(S_1) every associated prime of R is minimal.

(These are the first of Serre's conditions (R_n) and (S_n), see (23.15); compare Serre's criterion from Lecture 16: a Noetherian domain is normal if and only if it satisfies (R_1) and (S_2).)

(b) (Exercise 24.8) Let R be a Dedekind domain, and let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be ideals of R. By first reducing to the case that R is local, prove that

$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{c}),$$
$$\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c}).$$

(c) (Exercise 24.12) Prove that a semilocal Dedekind domain A is a PID. Begin by proving that each maximal ideal is principal. (A ring is *semilocal* if it has only finitely many maximal ideals.)

(d) (Exercise 24.13) Let R be a Dedekind domain, and let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero ideals of R. Prove that

(1) every ideal of $R/\mathfrak{a}$ is principal;

(2) $\mathfrak{b}$ is generated by two elements.

(e) (Exercise 22.4) Let R be a ring, I an ideal, and M an R-module. Prove that the following are equivalent:

(1) the canonical map from M to its I-adic completion $\widehat{M}$ is injective;

(2) $\bigcap_{n\geq 0} I^n M = 0$;

(3) M is Hausdorff in the I-adic topology.

Now assume that R is Noetherian and M is finitely generated, and that either (i) $I \subseteq \operatorname{rad}(R)$, or (ii) R is a domain, I is a proper ideal, and M is torsion-free. Prove that $M \to \widehat{M}$ is injective. (Here $\operatorname{rad}(R)$ is the Jacobson radical of R, the intersection of its maximal ideals, and M is torsion-free if $rm = 0$ implies $r = 0$ or $m = 0$, for $r \in R$ and $m \in M$.)

(f) (Exercise 22.10) Let A be a ring, let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be distinct maximal ideals of A, and let $I := \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$. Prove that the I-adic completion of A is isomorphic to the product of the completions of the local rings $A_{\mathfrak{m}_1}, \ldots, A_{\mathfrak{m}_r}$, that is,

$$\widehat{A}_I \simeq \widehat{A_{\mathfrak{m}_1}} \times \cdots \times \widehat{A_{\mathfrak{m}_r}}.$$

(For example, the (6)-adic completion of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$.)

(g) (Exercise 22.12)

(1) Let R be a Noetherian ring, I an ideal, and $\widehat{R}$ the I-adic completion of R. Prove that if $\mathrm{gr}_I(R)$ is a domain, then $\widehat{R}$ is a domain, and that if in addition $\bigcap_{n\geq 0} I^n = 0$, then R is a domain.

(2) Use (1) to give an alternative proof that a regular local ring is a domain.

(AK13 does not assume that R is Noetherian in (1); with this assumption you can use the isomorphism $\mathrm{gr}_{I\widehat{R}}(\widehat{R}) \simeq \mathrm{gr}_I(R)$ from Lectures 17–18. For (2) you may use Proposition (21.22): if A is a regular local ring with maximal ideal $\mathfrak{m}$, then $\mathrm{gr}_{\mathfrak{m}}(A)$ is a polynomial ring over $A/\mathfrak{m}$. Compare the local ring of the node from Lectures 17–18, a domain R for which neither $\mathrm{gr}_{\mathfrak{m}}(R)$ nor $\widehat{R}$ is a domain.)

(h) (Exercise 22.28, Nakayama's lemma for a complete ring) Let R be a ring, I an ideal, and M an R-module. Assume that R is I-adically complete and that M is I-adically separated. Prove that if the images of $m_1, \ldots, m_n \in M$ generate M/IM, then $m_1, \ldots, m_n$ generate M. (Note that M is not assumed to be finitely generated. The solution in AK13 appeals to the proof of Lemma (22.27), but you can also argue directly, by successive approximation.)

Remember to state which two of (a)–(h) you want graded (for example, by writing "Grade (c) and (g)" at the top of your solution to Problem 1), and to include the sentence "I checked my answer against the solution in AK13." under each of the others.

Problem 2. Survey (1 point)

Answer the following:

1. How long did it take you to complete this problem set (include everything)?
2. Do you think the pace of the lectures over the past two weeks was too slow, too fast, or just right?
3. Name one thing you liked and one thing you did not about this pset.
4. Name one thing you liked and one thing you did not about the lectures over the past two weeks.
5. Did you think the second midterm exam was too hard, too easy, or just right?