18.705 Commutative Algebra **Fall 2026**

Problem Set #5 **Due: 11/10/2026**

These problems are related to material covered in Lectures 12–15.

Instructions: The format of this problem set is the same as that of Problem Set 4. Problem 1 consists of eight exercises (a)–(h) from AK13 whose statements are reproduced below; solve all eight. Write up complete solutions to *at least two* of them, and clearly indicate which two you want graded (you are welcome to write up more than two, but only the two you indicate will be graded). For each exercise you do not write up, check your answer against the solution in AK13 (clicking the exercise number in AK13 will take you directly to the solution, be careful not to click prematurely) and, in place of a write-up, simply write the sentence "I checked my answer against the solution in AK13." Then complete the survey problem 2. Your answer should be in the form of a PDF submitted to Gradescope no later than midnight on the date due (do not ask for an extension, the answer is no). Be sure to include your name somewhere on the PDF. Each of the two graded exercises is worth 1 point and the grading is binary; any credible attempt will be awarded 1 point.

I recommend using latex to create the PDF, but scanned handwritten solutions are fine provided your handwriting is legible. For Problem 1 collaboration (including with LLMs) is permitted, but I strongly encourage you to first attempt the problems on your own. Problem 2 should be answered entirely on your own.

Important: Be sure to use the 2013 version of Altman–Kleiman; newer versions do not necessarily use the same numbering.

Problem 1. Problems from Altman–Kleiman (2 points)

Read Chapters 16, 19, and 21 of AK13, and the parts of Chapter 23 on discrete valuation rings: (23.1)–(23.3) and (23.10)–(23.14). Lectures 13–15 covered much of this material, in a different order. Note that AK13 derives Krull's principal ideal and height theorems from the Dimension Theorem (21.4), whose proof uses the Hilbert–Samuel polynomials of Chapter 20; you may skip Chapter 20 and the proofs of (21.3) and (21.4), and take (21.4) as given (for a Noetherian local ring A the equality $\dim(A) = s(A)$ in (21.4) is the corollary to Krull's height theorem proved in Lecture 14). You may also skip the material on depth and Cohen–Macaulay rings in Chapter 23 ((23.4)–(23.9) and the appendix); in the proof of Theorem (23.10) this affects only condition (3), which you may ignore.

(a) (Exercise 16.29) Let p be a prime number, and set $M := \mathbb{Z}[1/p]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Prove that any $\mathbb{Z}$-submodule $N \subset M$ is either finite or all of M. Deduce that M is an Artinian $\mathbb{Z}$-module, and that it is not Noetherian.

(b) (Exercise 19.8) Let k be a field, R a finitely generated k-algebra (an algebra-finite extension of k, in the terminology of AK13). Prove that R is Artinian if and only if R is a finite-dimensional k-vector space.

(c) (Exercise 21.16) Let R be a finitely generated algebra over a field. Assume R is a domain of dimension r. Let $x \in R$ be neither 0 nor a unit. Set $R' := R/\langle x \rangle$. Prove that every maximal chain of primes in R' (one that cannot be refined or extended) has length $r - 1$.

(d) (Exercise 21.11) Let R be a Noetherian ring, $\mathfrak{p}$ a prime of height at least 2. Prove that $\mathfrak{p}$ is the union of height-1 primes, but not of finitely many.

(e) (Exercise 21.14) Let R be a domain. Prove that, if R is a UFD, then every height-1 prime is principal, and that the converse holds if R is Noetherian.

(f) (Exercise 21.18) Let R be a Noetherian ring. Prove that

$$\dim(R[X]) = \dim(R) + 1.$$

(Compare Problem 1(g) of Problem Set 4, which shows that this can fail when R is not Noetherian; one approach is to bound the height of a prime P of $R[X]$ in terms of the height of $P \cap R$ using Krull's height theorem.)

(g) (Exercise 21.23) Let A be a Noetherian local ring of dimension r, and let $x_1, \ldots, x_s \in A$ with $s \leq r$. Set $\mathfrak{a} := \langle x_1, \ldots, x_s \rangle$ and $B := A/\mathfrak{a}$. Prove that the following are equivalent:

(1) A is regular, and there are $x_{s+1}, \ldots, x_r \in A$ such that $x_1, \ldots, x_r$ is a regular system of parameters;

(2) B is regular of dimension $r - s$.

(h) (Exercise 23.12) Let k be a field, $P := k[X, Y]$ the polynomial ring in two variables, $f \in P$ an irreducible polynomial. Say $f = \ell(X, Y) + g(X, Y)$ with $\ell(X, Y) = aX + bY$ for $a, b \in k$ and with $g \in \langle X, Y \rangle^2$. Set $R := P/\langle f \rangle$ and $\mathfrak{p} := \langle X, Y \rangle / \langle f \rangle$. Prove that $R_{\mathfrak{p}}$ is a DVR if and only if $\ell \neq 0$. (Thus $R_{\mathfrak{p}}$ is a DVR if and only if the plane curve $C\colon f = 0$ in k^2 is nonsingular at $(0, 0)$.)

Remember to state which two of (a)–(h) you want graded (for example, by writing "Grade (c) and (g)" at the top of your solution to Problem 1), and to include the sentence "I checked my answer against the solution in AK13." under each of the others.

Problem 2. Survey (1 point)

Answer the following:

1. How long did it take you to complete this problem set (include everything)?
2. Do you think the pace of the lectures over the past two weeks was too slow, too fast, or just right?
3. Name one thing you liked and one thing you did not about this pset.
4. Name one thing you liked and one thing you did not about the lectures over the past two weeks.
5. Is there a topic we have covered so far that you would like to see revisited, or a topic you hope to see covered in the remainder of the course?