

These problems are related to material covered in Lectures 9–12.

Instructions: The format of this problem set differs from the previous ones. Problem 1 consists of eight exercises (a)–(h) from [AK13](#) whose statements are reproduced below; solve all eight. Write up complete solutions to *at least two* of them, and clearly indicate which two you want graded (you are welcome to write up more than two, but only the two you indicate will be graded). For each exercise you do not write up, check your answer against the solution in [AK13](#) (clicking the exercise number in [AK13](#) will take you directly to the solution, be careful not to click prematurely) and, in place of a write-up, simply write the sentence “I checked my answer against the solution in AK13.” Then complete the survey problem 2. Your answer should be in the form of a PDF submitted to [Gradescope](#) no later than midnight on the date due (do not ask for an extension, the answer is no). Be sure to include your name somewhere on the PDF. Each of the two graded exercises is worth 1 point and the grading is binary; any credible attempt will be awarded 1 point.

I recommend using latex to create the PDF, but scanned handwritten solutions are fine provided your handwriting is legible. For Problem 1 collaboration (including with LLMs) is permitted, but I strongly encourage you to first attempt the problems on your own. Problem 2 should be answered entirely on your own.

Important: Be sure to use the [2013 version of Altman–Kleiman](#); newer versions do not necessarily use the same numbering.

Problem 1. Problems from Altman–Kleiman (2 points)

Read Chapters 14, 15, 17, and 18 of [AK13](#), as well as the material on support in Chapter 13 (starting at (13.26)); Lectures 9–12 covered much of this material. You may skip the appendix on Jacobson rings at the end of Chapter 15.

- (a) (Exercise 17.7) Take $R := \mathbb{Z}$ and $M := \mathbb{Z}/(2) \oplus \mathbb{Z}$. Find $\text{Ass}(M)$ and find two submodules $L, N \subseteq M$ with $L + N = M$ but $\text{Ass}(L) \cup \text{Ass}(N) \subsetneq \text{Ass}(M)$.
- (b) (Exercise 17.16) Let R be a Noetherian ring, M a module, N a submodule, $x \in R$. Show that, if $x \notin \mathfrak{p}$ for any $\mathfrak{p} \in \text{Ass}(M/N)$, then $xM \cap N = xN$.
- (c) (Exercise 14.12) Let R be a reduced ring, Σ the set of minimal primes of R . First show that $|\Sigma| = 1$ if and only if R is a domain ([AK13 \(3.34\)](#)), in which case $\Sigma = \{(0)\}$ and the statements below are trivial. Then prove that the set of zero divisors of R is $\bigcup_{\mathfrak{p} \in \Sigma} \mathfrak{p}$ and that $R_{\mathfrak{p}} = \text{Frac}(R/\mathfrak{p})$ for any $\mathfrak{p} \in \Sigma$.
- (d) (Exercise 18.17) Let k be a field, $R := k[X, Y, Z]$ be the polynomial ring. Set $\mathfrak{a} := (XY, X - YZ)$, set $\mathfrak{q}_1 := (X, Z)$ and set $\mathfrak{q}_2 := (Y^2, X - YZ)$. Show that $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2$ and that this expression is a minimal primary decomposition.
- (e) (Exercise 18.27) Let R be a Noetherian ring, $\mathfrak{p}$ a prime. Its *nth symbolic power* is $\mathfrak{p}^{(n)} := \mathfrak{p}^n R_{\mathfrak{p}} \cap R$, the saturation of $\mathfrak{p}^n$ with respect to $R \setminus \mathfrak{p}$.

- (1) Show $\mathfrak{p}^{(n)}$ is the $\mathfrak{p}$ -primary component of $\mathfrak{p}^n$.
 - (2) Show $\mathfrak{p}^{(m+n)}$ is the $\mathfrak{p}$ -primary component of $\mathfrak{p}^{(n)}\mathfrak{p}^{(m)}$.
 - (3) Show $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if and only if $\mathfrak{p}^n$ is $\mathfrak{p}$ -primary.
 - (4) Given a $\mathfrak{p}$ -primary ideal $\mathfrak{q}$, show $\mathfrak{q} \supseteq \mathfrak{p}^{(n)}$ for all large n .
- (f) (Exercise 14.5) Let $R \subseteq R'$ be an integral extension of domains, and $\mathfrak{p}$ a prime of R . Suppose R' has at least two distinct primes $\mathfrak{p}'$ and $\mathfrak{q}'$ lying over $\mathfrak{p}$. Show that $R'_{\mathfrak{p}'}$ is not integral over $R_{\mathfrak{p}}$. Show that, in fact, if y lies in $\mathfrak{q}'$, but not in $\mathfrak{p}'$, then $1/y \in R'_{\mathfrak{p}'}$ is not integral over $R_{\mathfrak{p}}$.
- (g) (Exercise 15.19) Let R be a ring, $R[X]$ the polynomial ring. Prove
- $$1 + \dim(R) \leq \dim(R[X]) \leq 1 + 2 \dim(R).$$
- (In particular, $\dim(R[X]) = \infty$ if and only if $\dim(R) = \infty$.)
- (h) (Exercise 15.17) Let k be a field, R a finitely generated k -algebra, $f \in R$ nonzero. Assume R is a domain. Prove that $\dim(R) = \dim(R_f)$.

Remember to state which two of (a)–(h) you want graded (for example, by writing “Grade (c) and (g)” at the top of your solution to Problem 1), and to include the sentence “I checked my answer against the solution in AK13.” under each of the others.

Problem 2. Survey (1 point)

Answer the following:

1. How long did it take you to complete this problem set (include everything)?
2. Do you think the pace of the lectures this week was too slow, too fast, or just right?
3. Name one thing you liked and one thing you did not about this pset.
4. Name one thing you liked and one thing you did not about the lectures this week.
5. Did you think the first midterm exam was too hard, too easy, or just right?