

These problems are related to material covered in Lectures 6–8.

Instructions: Solve problems 1 and 2 below, then complete the survey problem 3. Your answer should be in the form of a PDF submitted to [Gradescope](#) no later than midnight on the date due (do not ask for an extension, the answer is no). Be sure to include your name somewhere on the PDF. Each problem is worth 1 point and the grading is binary; unless otherwise specified, any credible attempt will be awarded 1 point.

I recommend using latex to create the PDF, but scanned handwritten solutions are fine provided your handwriting is legible. For the first two problems collaboration (including with LLMs) is permitted, but I strongly encourage you to first attempt the problems on your own. Problem 3 should be answered entirely on your own.

Important: Be sure to use the [2013 version of Altman–Kleiman](#); newer versions do not necessarily use the same numbering.

Problem 1. Self-graded problems from Altman–Kleiman (1 point)

Read chapters 5, 6, 8, and 9 of [AK13](#). Lectures 6–8 covered much of the material in chapters 5, 8, and 9, and the first part of chapter 6 formalizes the categorical language (functors, natural transformations, adjoint functors) that we used in Lecture 6; you may skip the material on direct limits in chapters 6 and 7, which is needed only for the proof of Lazard’s Theorem (9.24). Then solve the following 8 exercises in [AK13](#) and check your answers against the solution (clicking the exercise number in [AK13](#) will take you directly to the solution, be careful not to click prematurely).

4.19, 5.29, 6.5, 8.4, 8.12, 8.16, 9.15, 9.28

Do not include your answers in your Gradescope submission. Your answer to this problem should instead be a list of 8 positive integers, each counting the number of lines in the solution to the 8 problems above that is printed in [AK13](#).

Problem 2. The ideal and equational criteria for flatness (1 point)

Let R be a ring and M an R -module. Recall that M is *flat* if $N' \otimes_R M \rightarrow N \otimes_R M$ is injective for every inclusion of R -modules $N' \subseteq N$, and that a relation $\sum_{i=1}^n r_i x_i = 0$ with $r_i \in R$ and $x_i \in M$ is *trivial* if there are $y_1, \dots, y_m \in M$ and $a_{ij} \in R$ such that

$$x_i = \sum_{j=1}^m a_{ij} y_j \quad (1 \leq i \leq n) \quad \text{and} \quad \sum_{i=1}^n a_{ij} r_i = 0 \quad (1 \leq j \leq m).$$

Prove the following statements (you can find proofs in [AK13](#) and in the [Stacks Project](#), but I encourage you to work them out yourself; the hints below outline one route).

- (a) M is flat if and only if for every finitely generated ideal $\mathfrak{a} \subseteq R$ the map $\mathfrak{a} \otimes_R M \rightarrow M$ induced by the inclusion $\mathfrak{a} \hookrightarrow R$ is injective (equivalently, the map $\mathfrak{a} \otimes_R M \rightarrow \mathfrak{a}M$ defined by $r \otimes x \mapsto rx$ is an isomorphism).

- (b) M is flat if and only if every relation in M is trivial.
- (c) Every relation in M is trivial if and only if M satisfies the following condition: for every R -linear map $\alpha: R^n \rightarrow M$ and every $k \in \ker \alpha$ there is a factorization $\alpha = \beta \circ \varphi$ with $\varphi: R^n \rightarrow R^m$ and $\beta: R^m \rightarrow M$ linear and $\varphi(k) = 0$.

Hints. For (a), only the “if” direction requires proof. (i) Show that $\mathfrak{a} \otimes_R M \rightarrow M$ is injective for *every* ideal $\mathfrak{a}$: each element of $\mathfrak{a} \otimes_R M$ is the image of an element of $\mathfrak{a}_0 \otimes_R M$ for some finitely generated ideal $\mathfrak{a}_0 \subseteq \mathfrak{a}$. (ii) Suppose $N = N' + Rz$ and set $I := \{r \in R : rz \in N'\}$. Tensor the exact sequence $0 \rightarrow I \rightarrow N' \oplus R \rightarrow N \rightarrow 0$, in which $r \mapsto (-rz, r)$ and $(n', r) \mapsto n' + rz$, with M and use (i) to show that $N' \otimes_R M \rightarrow N \otimes_R M$ is injective. (iii) Treat the case that N/N' is finitely generated by induction on the number of generators. (iv) Reduce the general case to (iii).

For (b), if M is flat and $\sum_i r_i x_i = 0$, tensor the exact sequence $K \rightarrow R^n \rightarrow \mathfrak{a} \rightarrow 0$ with M , where $\mathfrak{a} := (r_1, \dots, r_n)$ and K is the kernel of the map $R^n \rightarrow \mathfrak{a}$ defined by $e_i \mapsto r_i$; use (a) for the converse.

Problem 3. Survey (1 point)

Answer the following:

1. How long did it take you to complete this problem set (include everything)?
2. Do you think the pace of the lectures this week was too slow, too fast, or just right?
3. Name one thing you liked and one thing you did not about this pset.
4. Name one thing you liked and one thing you did not about the lectures this week.