

Midterm 1 Solutions

Problem 1.

For each of the following modules M over the ring R (K a field), indicate whether M is finitely generated, free, projective, flat, injective, by writing *yes* or *no* in each cell of the table. No justification is required.

	f.g.	free	projective	flat	injective
(a) $R = \mathbb{Z}, M = \mathbb{Z}$	yes	yes	yes	yes	no
(b) $R = \mathbb{Z}, M = \mathbb{Q}$	no	no	no	yes	yes
(c) $R = \mathbb{Z}, M = \mathbb{Z}/2\mathbb{Z}$	yes	no	no	no	no
(d) $R = \mathbb{Z}, M = \mathbb{Q}/\mathbb{Z}$	no	no	no	no	yes
(e) $R = \mathbb{Z}, M = \mathbb{Z}[1/2]$	no	no	no	yes	no
(f) $R = K[x], M = K(x)$	no	no	no	yes	yes
(g) $R = K[x, y], M = (x, y)$	yes	no	no	no	no
(h) $R = K[x, y], M = K[x, y]/(x)$	yes	no	no	no	no
(i) $R = \mathbb{Z}/6\mathbb{Z}, M = \mathbb{Z}/2\mathbb{Z}$	yes	no	yes	yes	yes

Solution. Over a PID, a module is injective if and only if it is divisible, and projective if and only if it is free (Lecture 7). $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, K(x)$ are divisible; $\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}[1/2]$ are not. $\mathbb{Q}, \mathbb{Z}[1/2], K(x)$ are localizations of R , hence flat (Lecture 7); they are not finitely generated and not free (any two elements are linearly dependent, and they are not cyclic), hence not projective. A flat module over a domain is torsion-free: if $xm = 0$ with $x \neq 0$ then $m \in \text{Ann}(x)M = 0$ (AK 9.28, also on PS3). So $\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}/\mathbb{Z}, K[x, y]/(x)$ are not flat, hence not projective and not free.

(g) $M = (x, y)$ is not free, since $y \cdot x - x \cdot y = 0$ means that a basis has at most one element, but M is not principal. It is not flat: tensoring the inclusion $M \hookrightarrow R$ with M gives the map $M \otimes_R M \rightarrow R \otimes_R M = M, a \otimes b \mapsto ab$, which kills $x \otimes y - y \otimes x$; but this element is nonzero, since its image in $(M/\mathfrak{m}M) \otimes_K (M/\mathfrak{m}M)$, with $\mathfrak{m} = (x, y)$, is $\bar{x} \otimes \bar{y} - \bar{y} \otimes \bar{x} \neq 0$ in $K^2 \otimes_K K^2$. Hence not projective. It is not injective: an injective module over a domain is divisible (by Baer's criterion applied to $(a) \subseteq R$), but $x \notin xM$. $K[x, y]/(x)$ is not injective because it is not divisible: $y \cdot K[x, y]/(x) \neq K[x, y]/(x)$.

(i) $\mathbb{Z}/6\mathbb{Z} = \{0, 3\} \oplus \{0, 2, 4\}$, so $\mathbb{Z}/2\mathbb{Z} \simeq \{0, 3\}$ is a direct summand of a free module: projective, hence flat; not free, since $\#M = 2$ is not a power of 6. It is injective by Baer's criterion: the ideals of $\mathbb{Z}/6\mathbb{Z}$ are $(0), (2) \simeq \mathbb{Z}/3\mathbb{Z}, (3) \simeq \mathbb{Z}/2\mathbb{Z}$ and (1) ; a homomorphism $(2) \rightarrow \mathbb{Z}/2\mathbb{Z}$ is zero, and a homomorphism $(3) \rightarrow \mathbb{Z}/2\mathbb{Z}$ extends to $\mathbb{Z}/6\mathbb{Z}$ via the quotient map.

Problem 2.

Let $R \subset R'$ be a ring extension with an R -linear retraction $\rho: R' \rightarrow R$ (so $\rho(r) = r$ for all $r \in R$). Assume R' is Noetherian, and prove R is too.

Solution. Let $\mathfrak{a} \subset R$ be an ideal. As R' is Noetherian, the ideal $\mathfrak{a}R'$ is finitely generated; since $\mathfrak{a}$ generates $\mathfrak{a}R'$, finitely many elements $a_1, \dots, a_n \in \mathfrak{a}$ generate $\mathfrak{a}R'$ (Lecture 1). Given $a \in \mathfrak{a}$, write $a = a_1x'_1 + \dots + a_nx'_n$ with $x'_i \in R'$; applying the R -linear map ρ gives $a = a_1\rho(x'_1) + \dots + a_n\rho(x'_n)$ with $\rho(x'_i) \in R$. Thus $\mathfrak{a} = (a_1, \dots, a_n)$ is finitely generated, so R is Noetherian.

(Alternatively: a chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ of ideals of R gives a chain $\mathfrak{a}_1 R' \subset \mathfrak{a}_2 R' \subset \cdots$ of ideals of R' , which stabilizes, and $\rho(\mathfrak{a}_i R') = \mathfrak{a}_i \rho(R') = \mathfrak{a}_i$, so the original chain stabilizes.)

Problem 3.

Let R be a ring and $f \in R$. Show that $R_f \simeq R[x]/(fx - 1)$ as R -algebras.

Solution. The R -algebra map $R[x] \rightarrow R_f$ sending $x \mapsto 1/f$ is surjective (r/f^n is the image of rx^n) and kills $fx - 1$, so it induces a surjective R -algebra map $\psi: R[x]/(fx - 1) \rightarrow R_f$. In $R[x]/(fx - 1)$ the image of f is a unit (with inverse $\bar{x}$), so by the universal property of localization (Lecture 4), the map $R \rightarrow R[x]/(fx - 1)$ factors through a ring map $\theta: R_f \rightarrow R[x]/(fx - 1)$, $r/f^n \mapsto r\bar{x}^n$. Then $\psi\theta(r/f^n) = r/f^n$, so $\psi\theta = \text{id}$, and $\theta\psi$ is an R -algebra endomorphism of $R[x]/(fx - 1)$ fixing the generator $\bar{x}$, hence the identity. So ψ and θ are inverse isomorphisms.

(Alternatively, check that $R[x]/(fx - 1)$ has the universal property of R_f : for any ring map $\varphi: R \rightarrow A$ with $\varphi(f) \in A^\times$ there is a unique R -algebra map $R[x]/(fx - 1) \rightarrow A$, namely $x \mapsto \varphi(f)^{-1}$.)

Problem 4.

Let R be a ring, $\mathfrak{p}$ a prime ideal. Show that the image of $\text{Spec}(R_{\mathfrak{p}})$ in $\text{Spec}(R)$ is the intersection of all open neighborhoods of $\mathfrak{p}$ in $\text{Spec}(R)$.

Solution. The primes of $R_{\mathfrak{p}}$ are the $\mathfrak{q}R_{\mathfrak{p}}$ with $\mathfrak{q} \subseteq \mathfrak{p}$ prime, and the image of $\mathfrak{q}R_{\mathfrak{p}}$ in $\text{Spec}(R)$ is $\mathfrak{q}$ (Lecture 4); so the image $X_{\mathfrak{p}}$ of $\text{Spec}(R_{\mathfrak{p}})$ is the set of primes $\mathfrak{q} \subseteq \mathfrak{p}$. If $\mathfrak{q} \subseteq \mathfrak{p}$ and $f \notin \mathfrak{p}$ then $f \notin \mathfrak{q}$; thus $X_{\mathfrak{p}} \subseteq D(f)$ for every basic open neighborhood $D(f)$ of $\mathfrak{p}$, and since every open neighborhood of $\mathfrak{p}$ contains some $D(f) \ni \mathfrak{p}$, $X_{\mathfrak{p}}$ lies in every open neighborhood of $\mathfrak{p}$. Conversely, if $\mathfrak{q} \not\subseteq \mathfrak{p}$, pick $g \in \mathfrak{q} \setminus \mathfrak{p}$: then $D(g)$ is an open neighborhood of $\mathfrak{p}$ not containing $\mathfrak{q}$, so $\mathfrak{q}$ is not in the intersection.

Problem 5.

Let R be a ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals, and M a module.

(1) Use (8.13) to show that $(R/\mathfrak{a}) \otimes M = M/\mathfrak{a}M$.

(2) Use (1) to show that $(R/\mathfrak{a}) \otimes (R/\mathfrak{b}) = R/(\mathfrak{a} + \mathfrak{b})$.

(Here (8.13) is the right exactness of $\otimes$: if $M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact, then so is $M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N \rightarrow 0$.)

Solution. (1) Tensoring the exact sequence $\mathfrak{a} \rightarrow R \rightarrow R/\mathfrak{a} \rightarrow 0$ with M gives an exact sequence $\mathfrak{a} \otimes M \rightarrow R \otimes M \rightarrow (R/\mathfrak{a}) \otimes M \rightarrow 0$. Under the isomorphism $R \otimes M \simeq M$, $x \otimes m \mapsto xm$, the image of $\mathfrak{a} \otimes M$ is $\mathfrak{a}M$, so $(R/\mathfrak{a}) \otimes M \simeq M/\mathfrak{a}M$.

(2) Apply (1) with $M = R/\mathfrak{b}$: $(R/\mathfrak{a}) \otimes (R/\mathfrak{b}) = (R/\mathfrak{b})/\mathfrak{a}(R/\mathfrak{b}) = (R/\mathfrak{b})/((\mathfrak{a} + \mathfrak{b})/\mathfrak{b}) = R/(\mathfrak{a} + \mathfrak{b})$.

Problem 6.

Let I, J be ideals of a ring R . Show that the sequence

$$0 \rightarrow R/(I \cap J) \xrightarrow{\bar{r} \mapsto (\bar{r}, \bar{r})} R/I \oplus R/J \xrightarrow{(\bar{r}, \bar{s}) \mapsto \overline{r-s}} R/(I + J) \rightarrow 0$$

of R -modules is exact. Deduce that if $I + J = R$ then $R/(I \cap J) \simeq R/I \times R/J$.

Solution. Both maps are well defined (if $r \in I \cap J$ then $\bar{r} = 0$ in R/I and R/J ; if $r \in I$ and $s \in J$ then $r - s \in I + J$) and R -linear. The first map is injective: $(\bar{r}, \bar{r}) = 0$ means $r \in I \cap J$. The second map is surjective: $\bar{t}$ is the image of $(\bar{t}, 0)$. The composition is

zero. If $(\bar{r}, \bar{s}) \mapsto 0$ then $r - s = i + j$ with $i \in I$ and $j \in J$; putting $t := r - i = s + j$ we have $\bar{t} = \bar{r}$ in R/I and $\bar{t} = \bar{s}$ in R/J , so $(\bar{r}, \bar{s}) = (\bar{t}, \bar{t})$ lies in the image of the first map. Hence the sequence is exact. If $I + J = R$ then $R/(I + J) = 0$, so the first map is an isomorphism $R/(I \cap J) \simeq R/I \oplus R/J = R/I \times R/J$ (and it is a ring isomorphism, since $\bar{r} \mapsto (\bar{r}, \bar{r})$ is a ring map: this is the Chinese remainder theorem).

Problem 7.

Let R be a ring. Using the adjunction $\text{Hom}_R(M \otimes_R X, Y) \simeq \text{Hom}_R(X, \text{Hom}_R(M, Y))$ (natural in X and Y), prove that if P and Q are projective R -modules, then $P \otimes_R Q$ is a projective R -module.

Solution. A module N is projective if and only if the functor $\text{Hom}_R(N, -)$ is exact (Lecture 7). Taking $M = P$ and $X = Q$ in the adjunction gives an isomorphism of functors

$$\text{Hom}_R(P \otimes_R Q, -) \simeq \text{Hom}_R(Q, \text{Hom}_R(P, -)),$$

and the right hand side is the composition of the functors $\text{Hom}_R(P, -)$ and $\text{Hom}_R(Q, -)$, both exact because P and Q are projective. A composition of exact functors is exact, so $\text{Hom}_R(P \otimes_R Q, -)$ is exact and $P \otimes_R Q$ is projective.

Problem 8.

Let $R := \prod_{n \geq 1} \mathbb{F}_2$ be the product of countably many copies of the field $\mathbb{F}_2$ (with coordinatewise addition and multiplication).

- (a) Show that R is not Noetherian.
- (b) Show that every element of R is idempotent, and that the only idempotents of a local ring are 0 and 1.
- (c) Deduce that $R_{\mathfrak{p}} \simeq \mathbb{F}_2$ for every prime ideal $\mathfrak{p}$ of R .

Solution. (a) For $N \geq 1$ let $I_N := \{a \in R : a_n = 0 \text{ for all } n > N\}$. Then $I_1 \subsetneq I_2 \subsetneq \cdots$ is an ascending chain of ideals that does not stabilize.

(b) $(a_n)_n^2 = (a_n^2)_n = (a_n)_n$ since $a_n^2 = a_n$ in $\mathbb{F}_2$. If $e^2 = e$ in a local ring A with maximal ideal $\mathfrak{m}$, then $e(1 - e) = 0$; since $e + (1 - e) = 1 \notin \mathfrak{m}$, not both of e and $1 - e$ lie in $\mathfrak{m}$, so one of them is a unit (Lecture 5), and then the other is 0.

(c) $R_{\mathfrak{p}}$ is a local ring (Lecture 4), and it is nonzero since $0 \in \mathfrak{p}$ means 0 is not in the multiplicative set $R \setminus \mathfrak{p}$. Every element of $R_{\mathfrak{p}}$ is idempotent: $(a/w)^2 = a^2/w^2 = a/w$. By (b), $R_{\mathfrak{p}} = \{0, 1\}$ with $0 \neq 1$, so $R_{\mathfrak{p}} \simeq \mathbb{F}_2$. (In particular every localization $R_{\mathfrak{p}}$ is Noetherian although R is not, so being Noetherian is not a local property.)