

18.705 Commutative Algebra

Lecture 9

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Minimal primes

We now consider three ways of decomposing ideals. The first is geometric: for an ideal I of a noetherian ring R , write $V(I)$ as a finite union of sets of the form $V(\mathfrak{p})$.

Definition

Minimal primes of an ideal I are minimal elements of $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$. We write $\operatorname{Min}(I)$ for the set of minimal primes of I .

Lemma

Every prime $\mathfrak{p} \supseteq I$ contains a minimal prime of I , so $\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \operatorname{Min}(I)} \mathfrak{p}$, and the nilpotent elements of R are the elements that lie in every minimal prime of R .

Proof: The intersection of a chain of primes containing I is a prime containing I (check!), so Zorn's lemma, applied to the primes between I and $\mathfrak{p}$ ordered by reverse inclusion, yields a minimal prime of I inside $\mathfrak{p}$. The rest follows.

Finitely many minimal primes

Theorem

Let R be a Noetherian ring and I an ideal of R .

1. $\sqrt{I} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ for some finite set of primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.
2. $V(I) = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_n)$; for an affine variety,
 $Z_K(I) = Z_K(\mathfrak{p}_1) \cup \cdots \cup Z_K(\mathfrak{p}_n)$.
3. If no $\mathfrak{p}_i$ contains another $\mathfrak{p}_j$ (discard any that do), then $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} = \text{Min}(I)$.
In particular, $\text{Min}(I)$ is finite.

Proof: (1) To the blackboard! (2) $V(I) = V(\sqrt{I})$ and $V(\mathfrak{p} \cap \mathfrak{p}') = V(\mathfrak{p}) \cup V(\mathfrak{p}')$.
(3) Every minimal $\mathfrak{q} \in V(I)$ contains $\sqrt{I}$, hence some $\mathfrak{p}_i$.

For noetherian R the closed subsets of $\text{Spec } R$ are the finite unions of sets $V(\mathfrak{p})$: we have $V(I) = \bigcup_{\mathfrak{p} \in \text{Min}(I)} V(\mathfrak{p})$. The $V(\mathfrak{p})$ with $\mathfrak{p} \in \text{Min}(I)$ are the **irreducible components** of $V(I)$ (the maximal closed subsets not unions of proper closed subsets).

On the blackboard: radical ideals are finite intersections of primes

Theorem. If R is noetherian, every radical ideal of R is an intersection of finitely many prime ideals.

Proof. Suppose not. Since R is noetherian, there is a radical ideal J that is maximal among the radical ideals that are not finite intersections of primes. Then $J \neq R$ (R is the empty intersection), and J is not prime, so there are $a, b \in R \setminus J$ with $ab \in J$. The radical ideals $J_a := \sqrt{J + (a)}$ and $J_b := \sqrt{J + (b)}$ properly contain J .

Claim: $J_a \cap J_b = J$. If $x \in J_a \cap J_b$ then $x^n \in J + (a)$ and $x^m \in J + (b)$ for some n, m , so

$$x^{n+m} \in (J + (a))(J + (b)) \subseteq J + (ab) = J,$$

and hence $x \in \sqrt{J} = J$. The reverse inclusion is clear.

If J_a and J_b are both proper ideals, each is a finite intersection of primes by the maximality of J , and then so is $J = J_a \cap J_b$, a contradiction. If $J_a = R$, then $J = J_a \cap J_b = J_b \ni b$, contradicting $b \notin J$; likewise if $J_b = R$. □

Remark. The same argument shows that in a Noetherian ring every ideal is a finite intersection of **irreducible** ideals (ideals that are not the intersection of two strictly larger ideals).

Presented on the blackboard; not part of the projected slides.

The support of a module

Definition

The **support** of an R -module M is $\text{Supp}(M) := \{\mathfrak{p} \in \text{Spec } R : M_{\mathfrak{p}} \neq 0\}$.

- $\text{Supp}(R/I) = V(I)$: $(R/I)_{\mathfrak{p}}$ is generated by $\bar{1}/1$, which is 0 iff $w \cdot 1 \in I$ for some $w \notin \mathfrak{p}$, iff $I \not\subseteq \mathfrak{p}$.
- $\text{Supp}(M)$ is **specialization-closed**: if $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in \text{Supp}(M)$ then $\mathfrak{q} \in \text{Supp}(M)$, since $M_{\mathfrak{q}} = 0$ would give $M_{\mathfrak{p}} = (M_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}} = 0$ (inverting more elements).

Proposition

If M is finitely generated then $\text{Supp}(M) = V(\text{ann}_R(M))$, so $\text{Supp}(M)$ is closed.

Proof: Write $M = \sum_i Rm_i$, so that $\text{ann}_R(M) = \bigcap_i \text{ann}_R(m_i)$. Then $M_{\mathfrak{p}} = 0$ iff $m_i/1 = 0$ in $M_{\mathfrak{p}}$ for all i , iff $\text{ann}_R(m_i) \not\subseteq \mathfrak{p}$ for all i , iff $\bigcap_i \text{ann}_R(m_i) \not\subseteq \mathfrak{p}$, since $\mathfrak{p}$ is prime.

Supports of modules that are not finitely generated

Finite generation is necessary:

- Let $R = K[x]$ and $M = R_x/R = \bigoplus_{i>0} Kx^{-i}$, where x acts by multiplication and kills x^{-1} . Every element of M is killed by a power of x , so $M_{\mathfrak{p}} = 0$ when $x \notin \mathfrak{p}$, while $x^{-1}/1 \neq 0$ in $M_{(x)}$ as $\text{ann}_R(x^{-1}) = (x)$. So $\text{Supp}(M) = \{(x)\}$, but $\text{ann}_R(M) \subseteq \bigcap_n \text{ann}_R(x^{-n}) = \bigcap_n (x^n) = 0$.
- Let $R = \mathbb{C}[x]$ and $M = \bigoplus_{n \in \mathbb{Z}} R/(x - n)$. Localization commutes with direct sums, so $\text{Supp}(M) = \bigcup_n \text{Supp}(R/(x - n)) = \{(x - n) : n \in \mathbb{Z}\}$, which is not closed, while $\text{ann}_R(M) = \bigcap_n (x - n) = 0$.

Definition

Minimal primes of a module M are minimal elements of $\text{Supp}(M)$, denoted $\text{Min}(M)$. For finitely generated M these are the minimal primes of $\text{ann}_R(M)$, so $\text{Min}(M)$ is finite if R is noetherian.

Associated primes

Definition

A prime $\mathfrak{p}$ is an **associated prime** of an R -module M if $\mathfrak{p} = \operatorname{ann}_R(m)$ for some $m \in M$, equivalently, if there is an injective R -linear map $R/\mathfrak{p} \hookrightarrow M$. We write $\operatorname{Ass}_R(M)$ for this set; the associated primes of an ideal I are those of the module R/I .

- $\operatorname{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$ for $\mathfrak{p}$ prime: $\operatorname{ann}_R(\bar{r}) = \{s \in R : rs \in \mathfrak{p}\} = \mathfrak{p}$ for every nonzero $\bar{r} \in R/\mathfrak{p}$.
- $\operatorname{Ass}(M) \subseteq \operatorname{Supp}(M)$: if $R/\mathfrak{p} \hookrightarrow M$ then $0 \neq \kappa(\mathfrak{p}) = (R/\mathfrak{p})_{\mathfrak{p}} \hookrightarrow M_{\mathfrak{p}}$, by exactness of localization.
- For $R = K[x, y]$ and $I = (x^2, xy)$: $\operatorname{ann}_R(\bar{y}) = (x)$ and $\operatorname{ann}_R(\bar{x}) = (x, y)$, so $(x), (x, y) \in \operatorname{Ass}(R/I)$ (and these are all, as we will see). Here $V(I) = V(x)$: the **embedded** prime $(x, y) \notin \operatorname{Min}(I)$ is invisible to Spec .
- Every $X \subseteq \operatorname{Spec} R$ is $\operatorname{Ass}(M)$ for $M = \bigoplus_{\mathfrak{p} \in X} R/\mathfrak{p}$ (check!): finiteness of $\operatorname{Ass}(M)$ needs finite generation.

Associated primes and zerodivisors

Lemma

Let R be noetherian and M an R -module. Then

1. $\text{Ass}(M) \neq \emptyset$ if and only if $M \neq 0$;
2. $\bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p} = \{r \in R : rm = 0 \text{ for some } 0 \neq m \in M\}$, the *zerodivisors* on M .

Proof: It suffices to show that $\text{ann}_R(m)$ lies in an associated prime for each $0 \neq m \in M$. As R is noetherian, some $I = \text{ann}_R(m')$ with $0 \neq m' \in Rm$ is maximal among such I and $\text{ann}_R(m) \subseteq I$. If $rs \in I$ and $s \notin I$, then $sm' \neq 0$ and $\text{ann}_R(sm') \supseteq I$, so $\text{ann}_R(sm') = I$ and $r \in \text{ann}_R(sm') = I$. Thus I is prime.

The noetherian hypothesis is necessary: let $R = \bigcup_{n \geq 1} \mathbb{C}[[x^{1/n}]]$ (Puiseux series with nonnegative exponents). Every nonzero element of R is a unit times x^a with $a \in \mathbb{Q}_{\geq 0}$, so the annihilators of nonzero elements of $R/(x)$ are the ideals $\text{ann}(x^a + (x)) = (x^{1-a})$ with $0 \leq a < 1$, none of which is prime: $(x^{(1-a)/2})^2 \in (x^{1-a})$ but $x^{(1-a)/2} \notin (x^{1-a})$. So $\text{Ass}(R/(x)) = \emptyset$ even though $R/(x) \neq 0$.

Prime filtrations

Lemma

If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is exact, then $\text{Ass}(L) \subseteq \text{Ass}(M) \subseteq \text{Ass}(L) \cup \text{Ass}(N)$.

Proof: The first inclusion is clear. If $\mathfrak{p} = \text{ann}_R(m) \in \text{Ass}(M) \setminus \text{Ass}(L)$, then every nonzero element of $Rm \simeq R/\mathfrak{p}$ has annihilator $\mathfrak{p}$, so $Rm \cap L = 0$, and Rm maps isomorphically onto its image in N : thus $R/\mathfrak{p} \hookrightarrow N$.

Theorem

Let R be noetherian and M a finitely generated R -module. Then M has a *prime filtration*: a chain $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ of submodules with $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ for primes $\mathfrak{p}_i$. In particular, $\text{Ass}(M) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is finite.

Proof: If $M \neq 0$ it has an associated prime $\mathfrak{p}_1$, giving $M_1 \simeq R/\mathfrak{p}_1 \hookrightarrow M$; if $M/M_1 \neq 0$ it has an associated prime $\mathfrak{p}_2$, giving $M_2/M_1 \simeq R/\mathfrak{p}_2 \hookrightarrow M/M_1$. etc.. The chain ends since M is noetherian. By the lemma, $\text{Ass}(M_i) \subseteq \text{Ass}(M_{i-1}) \cup \{\mathfrak{p}_i\}$.

Generic freeness (junior version)

Prime filtrations reduce statements about finitely generated modules to statements about the quotient domains $R/\mathfrak{p}$. A first example, to be generalized later:

Corollary

Let R be a noetherian domain and M a finitely generated R -module. Then there is a nonzero $f \in R$ such that M_f is a free R_f -module.

Proof: Take a prime filtration (M_i) of M with quotients $R/\mathfrak{p}_i$, and pick $0 \neq f \in \prod_{\mathfrak{p}_i \neq 0} \mathfrak{p}_i$ (a product of nonzero ideals of a domain is nonzero). If $\mathfrak{p}_i \neq 0$ then $(R/\mathfrak{p}_i)_f = 0$, because $f \in \mathfrak{p}_i$ acts on $R/\mathfrak{p}_i$ as zero and on $(R/\mathfrak{p}_i)_f$ as a unit; so $(M_i)_f = (M_{i-1})_f$. If $\mathfrak{p}_i = 0$ then $0 \rightarrow (M_{i-1})_f \rightarrow (M_i)_f \rightarrow R_f \rightarrow 0$ is exact and splits, as R_f is free, so $(M_i)_f \simeq (M_{i-1})_f \oplus R_f$. By induction on i , every $(M_i)_f$ is free.

Geometrically: a finitely generated module over a domain is free over the dense open subset $D(f)$ of $\operatorname{Spec} R$ with **generic rank** $\dim_{\operatorname{Frac}(R)}(M \otimes_R \operatorname{Frac}(R))$ of M equal to the number of i with $\mathfrak{p}_i = 0$.

Associated primes localize

Lemma

Let R be noetherian, $W \subseteq R$ a multiplicative set, and M an R -module. Then

$$\text{Ass}_{W^{-1}R}(W^{-1}M) = \{W^{-1}\mathfrak{p} : \mathfrak{p} \in \text{Ass}_R(M), \mathfrak{p} \cap W = \emptyset\}.$$

Proof: To the blackboard!

Corollary

If R is noetherian then $\text{Min}(M) \subseteq \text{Ass}(M)$ for every R -module M .

Proof: Let $\mathfrak{p} \in \text{Min}(M)$. Then $M_{\mathfrak{p}} \neq 0$, so $M_{\mathfrak{p}}$ has an associated prime $\mathfrak{q}R_{\mathfrak{p}}$, with $\mathfrak{q} \subseteq \mathfrak{p}$. As $\text{Ass} \subseteq \text{Supp}$, we have $M_{\mathfrak{q}} = (M_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}} \neq 0$, so $\mathfrak{q} \in \text{Supp}(M)$, and $\mathfrak{q} = \mathfrak{p}$ by minimality. Thus $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}(M_{\mathfrak{p}})$, and $\mathfrak{p} \in \text{Ass}(M)$ by the lemma.

For noetherian R we now have $\bigcap_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \text{Ass}(R)} \mathfrak{p} = \{\text{nilpotents}\}$ and $\bigcup_{\mathfrak{p} \in \text{Ass}(R)} \mathfrak{p} = \{\text{zerodivisors}\}$. We will interpret $\bigcup_{\mathfrak{p} \in \text{Min}(R)} \mathfrak{p}$ in Chapter 6.

On the blackboard: associated primes localize

Lemma. Let R be noetherian, W a multiplicative set, and M an R -module.

Then $\text{Ass}_{W^{-1}R}(W^{-1}M) = \{W^{-1}\mathfrak{p} : \mathfrak{p} \in \text{Ass}_R(M), \mathfrak{p} \cap W = \emptyset\}$.

Proof. ($\supseteq$) Let $\mathfrak{p} \in \text{Ass}_R(M)$ with $\mathfrak{p} \cap W = \emptyset$. Then $W^{-1}\mathfrak{p}$ is a prime of $W^{-1}R$ (Lecture 4), and localizing an injection $R/\mathfrak{p} \rightarrow M$ gives an injection $W^{-1}R/W^{-1}\mathfrak{p} \simeq W^{-1}(R/\mathfrak{p}) \rightarrow W^{-1}M$, by exactness of localization. So $W^{-1}\mathfrak{p} \in \text{Ass}(W^{-1}M)$.

($\subseteq$) Every prime of $W^{-1}R$ has the form $W^{-1}\mathfrak{p}$ with $\mathfrak{p}$ prime and $\mathfrak{p} \cap W = \emptyset$. Suppose $W^{-1}\mathfrak{p} \in \text{Ass}(W^{-1}M)$, so that there is an injection $i: W^{-1}(R/\mathfrak{p}) \simeq W^{-1}R/W^{-1}\mathfrak{p} \rightarrow W^{-1}M$. Since R is noetherian, $R/\mathfrak{p}$ is finitely presented, so Hom localizes (Lecture 8):

$$W^{-1}\text{Hom}_R(R/\mathfrak{p}, M) \simeq \text{Hom}_{W^{-1}R}(W^{-1}(R/\mathfrak{p}), W^{-1}M).$$

Hence $i = \frac{1}{w} W^{-1}(i_0)$ for some $i_0 \in \text{Hom}_R(R/\mathfrak{p}, M)$ and $w \in W$, and $W^{-1}(i_0) = wi$ is injective. Let $K := \ker(i_0) \subseteq R/\mathfrak{p}$. Then $W^{-1}K = \ker(W^{-1}(i_0)) = 0$, so every element of K is killed by some element of W . But $W \cap \mathfrak{p} = \emptyset$, so the elements of W are nonzerodivisors on $R/\mathfrak{p}$. Hence $K = 0$, i_0 is injective, and $\mathfrak{p} \in \text{Ass}_R(M)$. $\square$

Remark. The noetherian hypothesis is necessary; $\text{Ass}_R(M)$ may be empty while $\text{Ass}(M_{\mathfrak{p}})$ is not.

Presented on the blackboard; not part of the projected slides.

Prime avoidance

Lemma (prime avoidance)

Let $I_1, \dots, I_n, J$ be ideals of R , with all but at most two of the I_i prime. If $J \subseteq I_1 \cup \dots \cup I_n$, then $J \subseteq I_i$ for some i . If moreover R is $\mathbb{N}$ -graded, the I_i are homogeneous primes, and J is a homogeneous ideal generated in positive degrees with $J \not\subseteq I_i$ for all i , then $J \setminus \bigcup_i I_i$ contains a homogeneous element.

Proof: To the blackboard!

Corollary

Let R be noetherian, M a finitely generated R -module, and I an ideal of R . If every element of I is a zerodivisor on M , then $Im = 0$ for some $0 \neq m \in M$.

Proof: $I \subseteq \bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}$, a finite union, so $I \subseteq \mathfrak{p} = \text{ann}_R(m)$ for some $\mathfrak{p} \in \text{Ass}(M)$.

Equivalently: if $I \not\subseteq \mathfrak{p}$ for every $\mathfrak{p} \in \text{Ass}(M)$, then I contains a nonzerodivisor on M .

On the blackboard: prime avoidance

Lemma. Let $I_1, \dots, I_n, J$ be ideals with I_i prime for $i > 2$. If $J \not\subseteq I_i$ for all i , then $J \not\subseteq \bigcup_i I_i$. Graded version: if R is $\mathbb{N}$ -graded, the I_i are homogeneous primes, J is homogeneous and generated in positive degrees, and $J \not\subseteq I_i$ for all i , then some homogeneous element of J lies outside $\bigcup_i I_i$.

Proof. Induction on n ; for $n = 1$ there is nothing to show. For $n \geq 2$, the induction hypothesis gives, for each i , an element $a_i \in J \setminus \bigcup_{j \neq i} I_j$. If some $a_i \notin I_i$ we are done, so assume $a_i \in I_i$ for every i , and consider

$$a := a_n + a_1 a_2 \cdots a_{n-1} \in J.$$

If $a \in I_i$ for some $i < n$: since $a_i \in I_i$ we get $a_1 \cdots a_{n-1} \in I_i$, hence $a_n \in I_i$, contradicting the choice of a_n . If $a \in I_n$: since $a_n \in I_n$ we get $a_1 \cdots a_{n-1} \in I_n$. For $n = 2$ this says $a_1 \in I_2$, a contradiction; for $n > 2$ the ideal I_n is prime, so $a_i \in I_n$ for some $i < n$, a contradiction. Thus $a \in J \setminus \bigcup_i I_i$.

In the graded case, the induction hypothesis gives homogeneous a_i (for $n = 1$, some homogeneous generator of J lies outside I_1), all of positive degree. Put $d := \deg a_1 + \cdots + \deg a_{n-1} > 0$ and $e := \deg a_n > 0$, and use $a := a_n^d + (a_1 \cdots a_{n-1})^e$ instead: it is homogeneous of degree de , and since the I_i are prime, a_n^d and $(a_1 \cdots a_{n-1})^e$ lie in exactly the same I_i as a_n and $a_1 \cdots a_{n-1}$, so the argument above goes through. □

Remark. The positive-degree hypothesis holds automatically if R_0 is a field and $J \neq R$, but it cannot be dropped in general: in $\mathbb{Z}[x]$ the homogeneous ideal $J = (2, x)$ is not contained in either of the homogeneous primes (2) and (x) , but every homogeneous element of J is a multiple of 2 or of x .

Presented on the blackboard; not part of the projected slides.