

18.705 Commutative Algebra

Lecture 8

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Adapted from the [2018 lecture notes](#) of Jack Jeffries (§4.3-4.4)

October 6, 2026

Flatness

Definition

Let M be an R -module. A relation $\sum_{i=1}^n r_i x_i = 0$ in M is **trivial** if there exist $y_1, \dots, y_m \in M$ and $a_{ij} \in R$ with $x_i = \sum_j a_{ij} y_j$ for all i and $\sum_i a_{ij} r_i = 0$ for all j .

Definition

An R -module M is **flat** if $L \otimes_R M \rightarrow N \otimes_R M$ is injective for every $L \subseteq N$. Equivalently, $- \otimes_R M$ takes short exact sequences to short exact sequences. Equivalently, every R -relation in M is trivial.

Definition

M is **faithfully flat** if $L \rightarrow N$ is injective iff $L \otimes_R M \rightarrow N \otimes_R M$ is injective. Nonzero free modules are faithfully flat; R_f is faithfully flat over R only if $f \in R^\times$.

Ideals and flat base change

Lemma

For ideals $I, J \subseteq R$ there is a short exact sequence

$$0 \rightarrow R/(I \cap J) \xrightarrow{\bar{r} \mapsto (\bar{r}, \bar{r})} R/I \oplus R/J \xrightarrow{(\bar{r}, \bar{s}) \mapsto \overline{r-s}} R/(I + J) \rightarrow 0.$$

Proof: Check! (The second map is surjective, and its kernel is the image of the first.)

Proposition

Let $R \rightarrow S$ be flat and $I, J, I_1, \dots, I_t$ ideals of R , with J finitely generated. Then $(I_1 \cap \dots \cap I_t)S = I_1S \cap \dots \cap I_tS$, and $(I : J)S = (IS : JS)$ where $I : J := \{r \in R : rJ \subseteq I\}$.

Proof: For $t = 2$, tensor the lemma's sequence with S : $(I_1 \cap I_2)S$ is then the kernel of $S \rightarrow S/I_1S \oplus S/I_2S$, which is $I_1S \cap I_2S$; now induct on t .

For $J = (j)$, the left exact $0 \rightarrow I : J \rightarrow R \xrightarrow{j} R/I$ stays left exact after $\otimes_R S$, which says $(I : J)S = (IS : jS)$; in general $I : (j_1, \dots, j_t) = \bigcap_i (I : j_i)$.

Hom and flat base change

Proposition

Let S be a flat R -algebra, M a finitely presented R -module, and N any R -module. Then $s \otimes \varphi \mapsto s \cdot (S \otimes \varphi)$ defines an isomorphism

$$S \otimes_R \operatorname{Hom}_R(M, N) \xrightarrow{\sim} \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N).$$

Proof: True for $M = R^{\oplus a}$, both sides being $(S \otimes_R N)^{\oplus a}$; for general M , apply both sides to a finite presentation of M and compare the resulting left exact sequences.

Corollary

If R is Noetherian and M is finitely generated, then for every multiplicative set W $W^{-1} \operatorname{Hom}_R(M, N) \simeq \operatorname{Hom}_{W^{-1}R}(W^{-1}M, W^{-1}N)$: Hom commutes with localization.

Local properties

Definition

A property $\mathcal{P}$ of rings is **local** if the following biconditionals both hold:

$$\mathcal{P}(R) \iff \mathcal{P}(R_{\mathfrak{p}}) \text{ for all } \mathfrak{p} \in \operatorname{Spec}(R) \iff \mathcal{P}(R_{\mathfrak{m}}) \text{ for all } \mathfrak{m} \in \operatorname{Max}(R).$$

Likewise for properties of modules, elements, and morphisms.

Proposition

The following are local properties:

- (1) $x = 0$, for $x \in M$;
- (2) $M = 0$;
- (3) $N = M$, for $N \subseteq M$;
- (4) $M \hookrightarrow N$ is injective;
- (5) $M \twoheadrightarrow N$;
- (6) $r/s \in R$, for R a domain.

Proof: To the blackboard!

On the blackboard: local properties

Proposition. These properties are local: (1) $x = 0$, for $x \in M$; (2) $M = 0$; (3) $N = M$, for $N \subseteq M$; (4) α injective, (5) α surjective, for $\alpha: M \rightarrow N$; (6) $\alpha \in R$, for R a domain and $\alpha \in \text{Frac}(R)$.

Proof. Maximal ideals are prime, so it suffices to show that each property passes to the localization at every prime $\mathfrak{p}$ (clear for (1) and (2)), and that if it fails, then it fails at some maximal ideal $\mathfrak{m}$.

(1) If $x \neq 0$ then $\text{ann}_R(x) \neq R$ lies in a maximal ideal $\mathfrak{m}$. If $x/1 = 0$ in $M_{\mathfrak{m}}$, then $w x = 0$ for some $w \in R \setminus \mathfrak{m}$; but then $w \in \text{ann}_R(x) \subseteq \mathfrak{m}$, a contradiction. So $x/1 \neq 0$ in $M_{\mathfrak{m}}$.

(2) If M contains $x \neq 0$ then $x/1 \neq 0$ in $M_{\mathfrak{m}}$ for some $\mathfrak{m}$, by (1), so $M_{\mathfrak{m}} \neq 0$.

(3) $(M/N)_{\mathfrak{p}} \simeq M_{\mathfrak{p}}/N_{\mathfrak{p}}$, as localization is exact, so $N_{\mathfrak{p}} = M_{\mathfrak{p}}$ iff $(M/N)_{\mathfrak{p}} = 0$; apply (2) to M/N .

(4) Localizing the exact sequence $0 \rightarrow \ker(\phi) \rightarrow M \xrightarrow{\phi} N$ at $\mathfrak{p}$ shows that $\ker(\phi_{\mathfrak{p}}) = \ker(\phi)_{\mathfrak{p}}$, so $\phi_{\mathfrak{p}}$ is injective iff $\ker(\phi)_{\mathfrak{p}} = 0$; apply (2) to $\ker(\phi)$.

(5) Localizing $M \xrightarrow{\phi} N \rightarrow \text{coker}(\phi) \rightarrow 0$ shows that $\text{coker}(\phi_{\mathfrak{p}}) = \text{coker}(\phi)_{\mathfrak{p}}$; apply (2) to $\text{coker}(\phi)$.

(6) Let $\alpha = r/s$ with $r, s \in R$, and let $(s : r) := \{a \in R : ar \in sR\}$. If $\alpha \in R$ then $\alpha \in R_{\mathfrak{p}}$ for every $\mathfrak{p}$, as $R \subseteq R_{\mathfrak{p}} \subseteq \text{Frac}(R)$. Colons commute with flat base change, so for every $\mathfrak{p}$ we have

$$r/s \in R_{\mathfrak{p}} \iff r \in sR_{\mathfrak{p}} \iff (sR_{\mathfrak{p}} : rR_{\mathfrak{p}}) = R_{\mathfrak{p}} \iff (s : r)R_{\mathfrak{p}} = R_{\mathfrak{p}}.$$

$\alpha \notin R \implies r \notin sR \implies (s : r) \neq R$ lies in a max ideal $\mathfrak{m}$. So $(s : r)R_{\mathfrak{m}} \subseteq \mathfrak{m}R_{\mathfrak{m}} \neq R_{\mathfrak{m}}$ and $\alpha \notin R_{\mathfrak{m}}$. $\square$

Presented on the blackboard; not part of the projected slides.

More local properties, and a warning

Proposition

Reduced is a local property of rings, and normal is a local property of domains.

Proof: Reduced: if $0 \neq r \in R$ is nilpotent, take $\mathfrak{m} \supseteq \text{ann}_R(r)$;

then $r/1$ is a nonzero nilpotent in $R_{\mathfrak{m}}$. The other direction is similar.

Normal: localizations of a domain R are domains with the same fraction field.

If R is normal and $r/s \in \text{Frac}(R)$ is integral over $R_{\mathfrak{p}}$, clearing the denominators $w \notin \mathfrak{p}$ of the coefficients shows rw/s is integral over R , so $rw/s \in R$ and $r/s \in R_{\mathfrak{p}}$.

If R is not normal, some $r/s \notin R$ is integral over R , hence over each $R_{\mathfrak{m}}$; apply (6).

$\mathcal{P}$ is **preserved by localization** if $\mathcal{P}(R) \Rightarrow \mathcal{P}(R_{\mathfrak{p}})$ for all $\mathfrak{p}$. “Domain” and “Noetherian” are preserved by localization, but they are **not** local properties:

- $R = K \times K$ is not a domain, but $R_{\mathfrak{p}} \simeq K$ is, for both of its primes $\mathfrak{p}$;
- $R = \prod_{n \geq 1} \mathbb{F}_2$ is not Noetherian, but $R_{\mathfrak{p}} \simeq \mathbb{F}_2$ is for every prime $\mathfrak{p}$ (check!).

Projective = locally free = flat

Lemma

A projective module P is finitely presented if and only if it is finitely generated.

Theorem

For a finitely presented R -module P the following are equivalent:

- 1. P is projective;*
- 2. P is a direct summand of a free module of finite rank: $P \oplus Q \simeq R^n$;*
- 3. P is **locally free**: $R = (f_1, \dots, f_n)$ with each P_{f_i} a free R_{f_i} -module;*
- 4. $P_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module for every prime ideal $\mathfrak{p}$;*
- 5. $P_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$ -module for every maximal ideal $\mathfrak{m}$;*
- 6. P is flat.*

Proof: To the blackboard!

On the blackboard: projective = locally free = flat

Lemma. A projective module P is finitely presented iff it is finitely generated.

Proof. For the nontrivial direction, if P is finitely generated, we have a surjection $R^n \twoheadrightarrow P$, which splits because P is projective. Its kernel is thus a quotient of R^n , hence finitely generated, hence the image of a morphism $R^m \rightarrow R^n$, yielding a finite presentation $R^m \rightarrow R^n \rightarrow P \rightarrow 0$. $\square$

Theorem. For a finitely presented P : (i) P is projective $\Leftrightarrow$ (ii) $P \oplus Q \simeq R^n \Leftrightarrow$ (iii) P is locally free $\Leftrightarrow$ (iv) P_p is free for all prime $p \Leftrightarrow$ (v) P_m is free for all maximal $m \Leftrightarrow$ (vi) P is flat.

Proof. (i) $\Rightarrow$ (ii): a surjection $R^n \twoheadrightarrow P$ splits. (ii) $\Rightarrow$ (vi): free modules are flat, and direct summands of flat modules are flat. (vi) $\Rightarrow$ (iv): flatness is preserved by localization, and a finitely generated flat module over the local ring R_p is free. (iv) $\Rightarrow$ (v): trivial.

(v) $\Rightarrow$ (iii): Fix a maximal ideal m . Lift a basis of P_m to elements of P (clear denominators), giving $\alpha: R^n \rightarrow P$ with α_m an isomorphism. Its cokernel C is finitely generated with $C_m = 0$, so some $g \notin m$ kills C , and α_g is surjective. Its kernel K is finitely generated (P_g is finitely presented over R_g ; exercise) with $K_m = 0$, so likewise $K_{gh} = 0$ for some $h \notin m$; thus α_f is an isomorphism for $f := gh \notin m$. The f_m generate an ideal lying in no maximal ideal, so 1 is a (finite) R -linear combination of the f_m .

(iii) $\Rightarrow$ (i): Let $M \twoheadrightarrow N$ be surjective. As P is finitely presented and R_{f_i} is flat over R , $\text{Hom}_R(P, M)_{f_i} \simeq \text{Hom}_{R_{f_i}}(P_{f_i}, M_{f_i})$ (flat base change), and similarly for N ; as P_{f_i} is free, $\text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, N)$ is surjective after localizing at each f_i . Every maximal ideal m misses some f_i , and localizing at m factors through localizing at f_i , so the map is surjective at every m , hence surjective: surjectivity is a local property. $\square$

Presented on the blackboard; not part of the projected slides.