

18.705 Commutative Algebra

Lecture 7

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Restriction of scalars and its adjoints

Corollary

Let $\varphi: A \rightarrow B$ be a ring map and $F: B\text{-mod} \rightarrow A\text{-mod}$ *restriction of scalars* (Y viewed as an A -module via φ). Then $(B \otimes_A -, F)$ and $(F, \text{Hom}_A(B, -))$ are adjoint pairs: for all A -modules X and B -modules Y ,

$$\begin{aligned}\text{Hom}_B(B \otimes_A X, Y) &\simeq \text{Hom}_A(X, FY), \\ \text{Hom}_A(FY, X) &\simeq \text{Hom}_B(Y, \text{Hom}_A(B, X)).\end{aligned}$$

Proof: B is an (A, B) -bimodule via φ . The theorem with A and B swapped says $B \otimes_A -$ is left adjoint to $\text{Hom}_B(B, -) \simeq F$ (a B -linear map $B \rightarrow Y$ is determined by the image of 1). The theorem says $F \simeq B \otimes_B -$ is left adjoint to $\text{Hom}_A(B, -)$.

Base change and restriction of scalars

For a ring map $\varphi: A \rightarrow B$ we now have three functors:

$$\begin{array}{ccc} & B \otimes_A - & \\ \swarrow & & \searrow \\ \{B\text{-modules}\} & \xrightarrow{F} & \{A\text{-modules}\} \\ \nwarrow & & \nearrow \\ & \text{Hom}_A(B, -) & \end{array}$$

Restriction of scalars F has left adjoint $B \otimes_A -$ (extension of scalars, or base change) and right adjoint $\text{Hom}_A(B, -)$ (coinduction). The unit and counit of $(B \otimes_A -, F)$ are $X \rightarrow F(B \otimes_A X)$, $x \mapsto 1 \otimes x$, and $B \otimes_A FY \rightarrow Y$, $b \otimes y \mapsto by$.

- $B \otimes_A A^{\oplus \Lambda} \simeq B^{\oplus \Lambda}$: base change takes free modules to free modules.
- For $A \rightarrow A/I$: $A/I \otimes_A M \simeq M/IM$ and $\text{Hom}_A(A/I, M) \simeq \text{ann}_M(I)$.
- For $A \rightarrow W^{-1}A$: $W^{-1}A \otimes_A M \simeq W^{-1}M$ (see the flatness slide).
- For a field extension $K \subseteq L$: $L \otimes_K V$ is the L -vector space on a basis of V .

Tensor products are right exact

Theorem

If (L, R) is an adjoint pair of functors between module categories, then L is right exact (it takes right exact sequences to right exact sequences) and R is left exact.

Proof: Let $\mathcal{S}: M' \rightarrow M \rightarrow M'' \rightarrow 0$ be right exact and Y a module. By naturality, $\text{Hom}(-, Y)$ applied to $L\mathcal{S}$ is $\text{Hom}(-, RY)$ applied to $\mathcal{S}$, which is left exact; As Y is arbitrary, $L\mathcal{S}$ is right exact by (2) “ $\Leftarrow$ ” of the Hom theorem. Dually for R .

Corollary

If $L \rightarrow M \rightarrow N \rightarrow 0$ is right exact then so is $L \otimes_R X \rightarrow M \otimes_R X \rightarrow N \otimes_R X \rightarrow 0$, for every R -module X . In particular, extension of scalars along $R \rightarrow S$ turns a presentation $R^m \xrightarrow{A} R^n \rightarrow M \rightarrow 0$ into a presentation $S^m \xrightarrow{A} S^n \rightarrow S \otimes_R M \rightarrow 0$.

– $\otimes_R X$ is not left exact: $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is injective, but $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{2} \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}$ is zero.

Projective modules

Theorem

For an R -module P the following are equivalent; P is *projective* if they hold.

1. Every short exact sequence $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ is *split*: $N \rightarrow P$ has a section (equivalently, $M \rightarrow N$ has a retraction; equivalently, $N \simeq M \oplus P$ compatibly).
2. Every surjection $N \twoheadrightarrow P$ is split, i.e. has a section $P \rightarrow N$.
3. The left exact functor $\text{Hom}_R(P, -)$ is exact.
4. For every surjection $\pi: M \twoheadrightarrow N$, every map $P \rightarrow N$ lifts to M :

$$\begin{array}{ccc} & & P \\ & \nearrow \text{dotted} & \downarrow \\ M & \xrightarrow[\pi]{\twoheadrightarrow} & N \end{array}$$

5. P is a direct summand of a free module: $P \oplus Q \simeq R^{\oplus I}$ for some Q and set I .

Proof: To the blackboard!

On the blackboard: characterizations of projective modules

Theorem. For an R -module P the following are equivalent: (i) every short exact sequence $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ is split; (ii) every surjection $N \twoheadrightarrow P$ is split; (iii) $\text{Hom}_R(P, -)$ is exact; (iv) for every surjection $\pi: M \twoheadrightarrow N$, every map $P \rightarrow N$ factors through π ; (v) P is a direct summand of a free module.

Proof. (i) $\Leftrightarrow$ (ii): a surjection $\pi: N \rightarrow P$ sits in the short exact sequence $0 \rightarrow \ker \pi \rightarrow N \rightarrow P \rightarrow 0$, and both conditions say that π has a section s (then $N = \ker \pi \oplus s(P)$, and conversely a decomposition $N \simeq M \oplus P$ compatible with π gives a section).

(i) $\Rightarrow$ (v): choose generators of P , i.e. a surjection $F \rightarrow P$ from a free module F , and let Q be its kernel. By (i), $F \simeq P \oplus Q$.

(v) $\Rightarrow$ (iii): for a family of modules (M_i) , $\text{Hom}_R(\bigoplus_i M_i, -) = \prod_i \text{Hom}_R(M_i, -)$, so property (iii) holds for $\bigoplus_i M_i$ if and only if it holds for every M_i . It holds for $M_i = R$ (as $\text{Hom}_R(R, -)$ is the identity functor), hence for all free modules, hence for direct summands of free modules.

(iii) $\Rightarrow$ (iv): apply $\text{Hom}_R(P, -)$ to $M \rightarrow N \rightarrow 0$ to get a surjection $\text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, N)$: every map $P \rightarrow N$ is $\pi \circ \psi$ for some $\psi: P \rightarrow M$.

(iv) $\Rightarrow$ (ii): apply (iv) to the surjection $N \twoheadrightarrow P$ and the map $\text{id}_P: P \rightarrow P$; the lift is a section. □

Remark. Free modules are projective, by (v), but projective modules need not be free: see the next slide.

Presented on the blackboard; not part of the projected slides.

Examples of projective modules

- Free modules are projective. Over a PID or a local ring, every projective module is free (the local case is proved below); over a field, every module is free.
- $R = K \times K$: the ideal $P = K \times 0$ is projective ($P \oplus (0 \times K) = R$) but not free, since $(0, 1)$ kills it. Likewise $\mathbb{Z}/2$ is a projective $\mathbb{Z}/6$ -module.
- Let $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, the polynomial functions on the sphere S^2 , and $v: R^3 \rightarrow R$, $f \mapsto xf_x + yf_y + zf_z$; for a vector field f , $v(f)(p) = p \cdot f(p)$. As $v(x, y, z) = 1$, $R^3 = R \cdot (x, y, z) \oplus P$ with $P := \ker v$ the tangent vector fields, so P is projective. It is not free: every tangent field on S^2 vanishes somewhere (hairy ball theorem), and an R -linear $P \rightarrow R$ is $f \mapsto g \cdot f$ with $g \in R^3$, so it is never surjective, whereas a nonzero free module surjects onto R .
- Nonprincipal ideals of Dedekind domains, e.g. $(2, 1 + \sqrt{-5}) \subseteq \mathbb{Z}[\sqrt{-5}]$ (18.785).

Injective modules

Theorem

For an R -module I the following are equivalent; I is *injective* if they hold.

1. Every short exact sequence $0 \rightarrow I \rightarrow M \rightarrow N \rightarrow 0$ is split.
2. Every injection $I \hookrightarrow M$ is split, i.e. has a retraction $M \rightarrow I$.
3. The left exact contravariant functor $\text{Hom}_R(-, I)$ is exact.
4. For every injection $\iota: M \hookrightarrow N$, every map $M \rightarrow I$ extends to N :

$$\begin{array}{ccc} M & \xhookrightarrow{\iota} & N \\ \downarrow & \nearrow & \\ I & \hookrightarrow & \end{array}$$

5. (*Baer's criterion*) As (iv), but only for the inclusions $J \hookrightarrow R$ of ideals of R .

Proof: (i)–(iv): reverse the arrows in the proof for projective modules.

(iv) $\Leftrightarrow$ (v): exercise (Zorn's lemma).

If R is a PID, I is injective iff it is *divisible* ($I = rI$ for all $r \neq 0$), by Baer:
a map $(a) \rightarrow I$ is $ra \mapsto rx$ for some $x \in I$, and it extends to R iff $x \in aI$.
So $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$ -modules, but $\mathbb{Z}$ is not.

Flat modules

Definition

An R -module N is **flat** if $M' \otimes_R N \rightarrow M \otimes_R N$ is injective for every $M' \subseteq M$, equivalently, $- \otimes_R N$ takes short exact sequences to short exact sequences. A ring homomorphism $R \rightarrow S$ is **flat** if S is a flat R -module.

- Projective modules (in particular free ones) are flat: exercise, reduce to $N = R$.
- Base change: if M is flat over R and $R \rightarrow S$ is a ring map, then $S \otimes_R M$ is flat over S , since $(S \otimes_R M) \otimes_S N \simeq M \otimes_R N$.
- Localization is flat: for any multiplicative set $W \subseteq R$ and R -module M we have $W^{-1}R \otimes_R M \simeq W^{-1}M$ (via $r/w \otimes m \mapsto rm/w$). The ring homomorphism $R \rightarrow W^{-1}R$ is flat, since the functor $W^{-1}(-)$ preserves injections (see Lecture 4).

Equational criterion

There is an alternative definition of flatness that is easier to apply in many contexts.

Definition

Let M be an R -module. A relation $\sum_i r_i x_i = 0$ in M is **trivial** if there are $y_j \in M$ and $a_{ij} \in R$, such that $x_i = \sum_j a_{ij} y_j$ for $1 \leq i \leq n$ and $\sum_i a_{ij} r_i = 0$ for $1 \leq j \leq m$.

Theorem (equational criterion)

An R -module M is flat if and only if every relation in M is trivial.

Proof: See Problem Set 3.

Finitely generated flat modules over local rings are free

Proof of the equational criterion:

Theorem

Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R -module. Then M is flat iff M is free; more precisely, if M is flat then every minimal generating set of M is a basis. 1

Proof: To the blackboard!

Similarly, if R is $\mathbb{N}$ -graded with R_0 a field, f.g. graded R -modules are flat iff free.

On the blackboard: f.g. flat modules over local rings are free

Theorem. Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated flat R -module. Then every minimal generating set of M is an R -basis for M , showing that M is free.

Proof. It suffices to show: if $x_1, \dots, x_n \in M$ have k -linearly independent images in $M/\mathfrak{m}M$, then $x_1, \dots, x_n$ are R -linearly independent. (Applied to a minimal generating set, which generates M by NAK, this gives a basis.) We proceed by induction on n . The base case $n = 0$ is trivial.

$n \geq 1$: Suppose $\sum_{i=1}^n r_i x_i = 0$. By the equational criterion, there exist $y_j \in M$ and $a_{ij} \in R$ such that $x_i = \sum_j a_{ij} y_j$ for each i , with $\sum_i a_{ij} r_i = 0$ for every j . Since $x_n \notin \mathfrak{m}M$, some $a_{nj} \notin \mathfrak{m}$ is a unit, and solving the j th relation gives

$$r_n = - \sum_{i=1}^{n-1} \frac{a_{ij}}{a_{nj}} r_i.$$

Substituting into $\sum_i r_i x_i = 0$ yields $\sum_{i=1}^{n-1} r_i (x_i - \frac{a_{ij}}{a_{nj}} x_n) = 0$. The images in $M/\mathfrak{m}M$ of the $n-1$ elements $x_i - \frac{a_{ij}}{a_{nj}} x_n$ are linearly independent (a relation among them is a relation among the x_i with the same coefficients on $x_1, \dots, x_{n-1}$), so by induction $r_1 = \dots = r_{n-1} = 0$, and then $r_n = 0$ by the displayed formula. $\square$

Remark. For an $\mathbb{N}$ -graded ring R with R_0 a field and a finitely generated $\mathbb{Z}$ -graded module M , the same argument with homogeneous x_i and R_+ in place of $\mathfrak{m}$ shows that M is flat iff it is free.

Presented on the blackboard; not part of the projected slides.