

18.705 Commutative Algebra

Lecture 6

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Exact sequences

Definition

A sequence $\cdots \rightarrow M_{i-1} \xrightarrow{\alpha_{i-1}} M_i \xrightarrow{\alpha_i} M_{i+1} \rightarrow \cdots$ of R -linear maps is **exact** if $\ker(\alpha_i) = \operatorname{im}(\alpha_{i-1})$ at every interior term M_i .

Special cases, for $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$:

- $0 \rightarrow L \rightarrow M \rightarrow N$ is **left exact** if α injective and $\ker \beta = \operatorname{im} \alpha$ ($L \simeq \ker \beta$);
- $L \rightarrow M \rightarrow N \rightarrow 0$ is **right exact** if β surjective and $\ker \beta = \operatorname{im} \alpha$ ($N \simeq \operatorname{coker} \alpha$);
- $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is **short exact** if both hold ($L \simeq \alpha(L)$ and $N \simeq M/\alpha(L)$).

Definition

A **presentation** of N is a right exact $F_1 \xrightarrow{\alpha} F_0 \rightarrow N \rightarrow 0$ with F_0, F_1 free: the basis of F_0 gives generators of N , and $\operatorname{im} \alpha$ the relations among them. N is **finitely presented** if F_0 and F_1 can be taken finitely generated.

Every module has a presentation; over a Noetherian ring, f.g. = finitely presented.

The Hom functors

$\text{Hom}_R(M, N) := \{R\text{-linear } \varphi: M \rightarrow N\}$ is an R -module: $(r\varphi)(m) := r\varphi(m)$.

Each $\alpha \in \text{Hom}_R(M, N)$ induces, for every R -module L ,

$$\alpha_*: \text{Hom}_R(L, M) \rightarrow \text{Hom}_R(L, N), \quad \varphi \mapsto \alpha \circ \varphi,$$

$$\alpha^*: \text{Hom}_R(N, L) \rightarrow \text{Hom}_R(M, L), \quad \varphi \mapsto \varphi \circ \alpha.$$

So $\text{Hom}_R(L, -)$ is a **covariant functor** and $\text{Hom}_R(-, L)$ is a **contravariant functor**: both preserve identities and compositions; the second reverses the direction of arrows.

Example

- $\text{Hom}_R(R, M) \simeq M$ via $\varphi \mapsto \varphi(1)$; under this identification α_* becomes α .
- $\text{Hom}_R(R/I, M) \simeq \text{ann}_M(I) := \{m \in M : Im = 0\}$, again via $\varphi \mapsto \varphi(1)$; so $\text{Hom}_R(R/I, -)$ is the functor $M \mapsto \text{ann}_M(I)$, $\alpha \mapsto \alpha|_{\text{ann}_M(I)}$.

Hom is left exact

Theorem

1. $0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is left exact iff for every R -module X the sequence $0 \rightarrow \operatorname{Hom}_R(X, L) \xrightarrow{\alpha_*} \operatorname{Hom}_R(X, M) \xrightarrow{\beta_*} \operatorname{Hom}_R(X, N)$ is left exact.
2. $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$ is right exact iff for every R -module X the sequence $0 \rightarrow \operatorname{Hom}_R(N, X) \xrightarrow{\beta^*} \operatorname{Hom}_R(M, X) \xrightarrow{\alpha^*} \operatorname{Hom}_R(L, X)$ is left exact.

Proof: To the blackboard!

So $\operatorname{Hom}_R(X, -)$ preserves kernels, and $\operatorname{Hom}_R(-, X)$ turns cokernels into kernels.

But Hom need not preserve surjectivity: applying $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ to $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$ gives $0 \rightarrow \mathbb{Z}/2$, and applying $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2)$ to $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ gives $\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$.

On the blackboard: Hom is left exact

Theorem. (1) $0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is left exact iff $0 \rightarrow \operatorname{Hom}(X, L) \xrightarrow{\alpha_*} \operatorname{Hom}(X, M) \xrightarrow{\beta_*} \operatorname{Hom}(X, N)$ is left exact for all X . (2) $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$ is right exact iff $0 \rightarrow \operatorname{Hom}(N, X) \xrightarrow{\beta^*} \operatorname{Hom}(M, X) \xrightarrow{\alpha^*} \operatorname{Hom}(L, X)$ is left exact for all X .

Proof. (1, $\Rightarrow$) Let $0 \rightarrow L \rightarrow M \rightarrow N$ be left exact and X a module. α_* is injective: if $\varphi: X \rightarrow L$ is nonzero then so is $\alpha \circ \varphi$, since α is injective. $\ker(\beta_*) = \operatorname{im}(\alpha_*)$: $\beta \circ \varphi = 0$ iff $\operatorname{im} \varphi \subseteq \ker \beta = \operatorname{im} \alpha$ iff $\varphi = \alpha \circ \psi$ for a (unique) $\psi: X \rightarrow L$, i.e. $\varphi \in \operatorname{im}(\alpha_*)$. (1, $\Leftarrow$) Take $X = R$: as $\operatorname{Hom}_R(R, -)$ is the identity functor, the sequence $0 \rightarrow L \rightarrow M \rightarrow N$ itself is left exact.

(2, $\Rightarrow$) Let $L \rightarrow M \rightarrow N \rightarrow 0$ be right exact and X a module. β^* is injective: if $\varphi: N \rightarrow X$ is nonzero, pick n with $\varphi(n) \neq 0$ and m with $\beta(m) = n$; then $(\varphi \circ \beta)(m) \neq 0$. $\ker(\alpha^*) = \operatorname{im}(\beta^*)$: $\varphi \circ \alpha = 0$ iff $\operatorname{im} \alpha \subseteq \ker \varphi$ iff φ factors through $M/\operatorname{im} \alpha \simeq N$, i.e. $\varphi = \psi \circ \beta$ with $\psi \in \operatorname{Hom}(N, X)$.

(2, $\Leftarrow$) Suppose the Hom sequence is left exact for every X . β is surjective: otherwise $X := N/\operatorname{im} \beta \neq 0$ and the projection $\varphi: N \rightarrow X$ is nonzero with $\beta^*(\varphi) = \varphi \circ \beta = 0$, contradicting the injectivity of β^* . $\operatorname{im} \alpha \subseteq \ker \beta$: take $X = N$ and $\varphi = \operatorname{id}_N$; then $\beta = \beta^*(\operatorname{id}_N) \in \ker(\alpha^*)$, i.e. $\beta \circ \alpha = 0$. $\ker \beta \subseteq \operatorname{im} \alpha$: take $X = M/\operatorname{im} \alpha$ and $\varphi: M \rightarrow X$ the projection; then $\alpha^*(\varphi) = \varphi \circ \alpha = 0$, so $\varphi = \psi \circ \beta$ for some $\psi: N \rightarrow X$, and hence $\ker \beta \subseteq \ker \varphi = \operatorname{im} \alpha$. □

Presented on the blackboard; not part of the projected slides.

Tensor products

Definition

The **tensor product** of R -modules M and N is the abelian group $M \otimes_R N$ generated by symbols $m \otimes n$ ($m \in M$, $n \in N$) subject to the relations

$$(m+m') \otimes n = m \otimes n + m' \otimes n, \quad m \otimes (n+n') = m \otimes n + m \otimes n', \quad (rm) \otimes n = m \otimes (rn);$$

it is an R -module via $r(m \otimes n) := (rm) \otimes n$; concretely, take the free abelian group on $M \times N$ modulo the relations. Elements are finite sums of **simple tensors** $m \otimes n$.

Proposition (universal property)

Every R -bilinear map of R -modules $\varphi: M \times N \rightarrow P$ factors through $M \otimes N$ uniquely via some R -module morphism $\phi: M \otimes N \rightarrow P$ for which $\varphi = \phi \circ ((m, n) \mapsto m \otimes n)$.

This determines $M \otimes_R N$ up to unique isomorphism. Check: $R \otimes_R M \simeq M$, $R/I \otimes_R M \simeq M/IM$, $M \otimes_R N \simeq N \otimes_R M$, and $\otimes_R$ commutes with direct sums.

Categories and functors

A (locally small) **category** $\mathcal{C}$ has objects X , sets of morphisms $\text{Hom}_{\mathcal{C}}(X, Y)$ with associative composition $\text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$, and identity morphisms that satisfy $f \circ \text{id}_X = f$ and $\text{id}_Y \circ f = f$ for all $f: X \rightarrow Y$.

$f: X \rightarrow Y$ is an **isomorphism** if $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$ for some $g: Y \rightarrow X$.

A **functor** $F: \mathcal{C} \rightarrow \mathcal{D}$ sends objects X to objects FX and morphisms $f: X \rightarrow Y$ to morphisms $Ff: FX \rightarrow FY$, with $F(\text{id}_X) = \text{id}_{FX}$ and $F(g \circ f) = Fg \circ Ff$.

A **contravariant functor** reverses arrows: $Ff: FY \rightarrow FX$ and $F(g \circ f) = Ff \circ Fg$; it is the same as a functor on the opposite category $\mathcal{C}^{\text{opp}}$.

Examples: forgetful functors (e.g. R -modules $\rightarrow$ sets); on R -modules, $\text{Hom}_R(L, -)$ and $- \otimes_R M$ (covariant) and $\text{Hom}_R(-, L)$ (contravariant); localization $W^{-1}(-)$ from R -modules to $W^{-1}R$ -modules; restriction of scalars along a ring homomorphism.

We work concretely with R -modules. Exact sequences, Hom, adjoint functors, and projective and injective objects make sense in any abelian category (which we will not define here); tensor products (and hence flatness) need extra structure.

Morphisms of functors

Definition

Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors. A **natural transformation** (or **morphism of functors**) $\eta: F \Rightarrow G$ is a family of morphisms $\eta_X: FX \rightarrow GX$, one for each object X of $\mathcal{C}$, such that for every morphism $f: X \rightarrow Y$ in $\mathcal{C}$ the square

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ Ff \downarrow & & \downarrow Gf \\ FY & \xrightarrow{\eta_Y} & GY \end{array}$$

commutes. If every η_X is an isomorphism, η is a **natural isomorphism** ($F \simeq G$).

The **functor category** $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and whose morphisms are natural transformations. Composition of natural transformations is associative, and for each $F \in \mathcal{D}^{\mathcal{C}}$ we have the identity natural transformation $\text{id}_F: F \rightarrow F$ defined by the family of identity morphisms id_{FX} for $X \in \mathcal{C}$.

Adjoint functors

Definition

Let $L: \mathcal{C} \rightarrow \mathcal{D}$ and $R: \mathcal{D} \rightarrow \mathcal{C}$ be functors. (L, R) is an **adjoint pair** (L is the **left adjoint** of R and R the **right adjoint** of L) if there are bijections $\text{Hom}_{\mathcal{D}}(LX, Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(X, RY)$ for $X \in \mathcal{C}$, $Y \in \mathcal{D}$, **naturally in X, Y** : for every $\alpha: X \rightarrow X'$ in $\mathcal{C}$ and $\beta: Y \rightarrow Y'$ in $\mathcal{D}$, the squares

$$\begin{array}{ccc} \text{Hom}(LX', Y) & \xrightarrow{\sim} & \text{Hom}(X', RY) \\ \text{compose with } L\alpha \downarrow & & \downarrow \text{compose with } \alpha \\ \text{Hom}(LX, Y) & \xrightarrow{\sim} & \text{Hom}(X, RY) \\ \text{Hom}(LX, Y) & \xrightarrow{\sim} & \text{Hom}(X, RY) \\ \text{compose with } \beta \downarrow & & \downarrow \text{compose with } R\beta \\ \text{Hom}(LX, Y') & \xrightarrow{\sim} & \text{Hom}(X, RY') \end{array}$$

commute.

Example: the free-module functor $S \mapsto R^{\oplus S}$ is left adjoint to the forgetful functor from R -modules to sets: $\text{Hom}_R(R^{\oplus S}, M) \simeq \text{Hom}_{\text{Sets}}(S, M)$.

The tensor–Hom adjunction

Theorem (tensor–Hom adjunction)

Let A, B be rings and M an (A, B) -bimodule (an A -module and a B -module with $a(bm) = b(am)$). Then $M \otimes_B -: B\text{-mod} \rightarrow A\text{-mod}$ (with A acting via M) and $\text{Hom}_A(M, -): A\text{-mod} \rightarrow B\text{-mod}$ (with $(b\varphi)(m) := \varphi(bm)$) form an adjoint pair:

$$\text{Hom}_A(M \otimes_B X, Y) \simeq \text{Hom}_B(X, \text{Hom}_A(M, Y)).$$

Proof: Call a biadditive $f: M \times X \rightarrow Y$ (A, B) -compatible if $f(am, x) = af(m, x)$ and $f(bm, x) = f(m, bx)$. Currying $f \mapsto (x \mapsto f(-, x))$ is a bijection from compatible maps to $\text{Hom}_B(X, \text{Hom}_A(M, Y))$: $f(-, x)$ is A -linear by the first condition, and $x \mapsto f(-, x)$ is B -linear by the second. By the universal property of $M \otimes_B X$, compatible maps also correspond to A -linear maps $M \otimes_B X \rightarrow Y$. Naturality: check!

For one ring R : $\text{Hom}_R(M \otimes_R X, Y) \simeq \text{Hom}_R(X, \text{Hom}_R(M, Y))$, naturally in X, Y .