

# 18.705 Commutative Algebra

## Lecture 4

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Adapted from the [2018 lecture notes](#) of Jack Jeffries (§2.4, §3.1-2)

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## In previous lectures...

### Theorem (Strong Nullstellensatz)

*Let  $K = \overline{K}$ , let  $I \subseteq K[x] := K[x_1, \dots, x_d]$  be an ideal, and let  $f \in K[x]$ . Then  $f$  vanishes on  $Z_K(I)$  if and only if  $f^n \in I$  for some  $n \geq 1$ .*

### Corollary

*Let  $K = \overline{K}$ . Then  $Z_K$  and  $I$  are inverse, inclusion-reversing bijections between radical ideals of  $K[x]$  and subvarieties of  $K^d$ ; so  $Z_K(I) = Z_K(J)$  iff  $\sqrt{I} = \sqrt{J}$ .*

### Definition

$\text{Spec}(R)$  is the set of primes of  $R$ , with closed sets  $V(I) := \{\mathfrak{p} : \mathfrak{p} \supseteq I\}$ ; the  $D(f) := \text{Spec}(R) \setminus V(f)$  form a basis; closed points = maximal ideals.  
 $\varphi: R \rightarrow S$  induces a continuous map  $\varphi^*: \text{Spec}(S) \rightarrow \text{Spec}(R)$ ,  $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$ .

# Projective space and homogeneous ideals

Recall that  $\mathbb{P}_K^n := (K^{n+1} \setminus \{0\})/\sim$ , where  $a \sim \lambda a$  for  $\lambda \in K^\times$ , is the set of lines through 0 in  $K^{n+1}$ . We write  $[a] = [a_0 : \cdots : a_n]$  for the class of  $a$ .

Let  $R := K[x_0, \dots, x_n]$  with the standard grading. If  $f \in R_d$  then  $f(\lambda a) = \lambda^d f(a)$ , so for homogeneous  $f$  the condition  $f(a) = 0$  depends only on  $[a]$ , unlike  $f(a)$  itself.

## Lemma

*Let  $I \subseteq R$  be a homogeneous ideal. If  $a \in Z_K(I)$  then  $\lambda a \in Z_K(I)$  for all  $\lambda \in K$ .*

**Proof:** Each  $f \in I$  has homogeneous components  $f_d \in I$ , and  $f_d(\lambda a) = \lambda^d f_d(a) = 0$ .

So  $Z_K(I)$  is a **cone** (a union of lines through 0), and it determines a subset

$$\overline{Z}_K(I) := \{[a] \in \mathbb{P}_K^n : a \in Z_K(I)\} \subseteq \mathbb{P}_K^n.$$

Subsets of this form are the **subvarieties** of  $\mathbb{P}_K^n$  (or **projective varieties**).

# The projective Nullstellensatz

For  $X \subseteq \mathbb{P}_K^n$  let  $I(X) \subseteq R$  be the ideal generated by the homogeneous  $f \in R$  with  $f(a) = 0$  for all  $[a] \in X$ . It is a homogeneous radical ideal (check!), and  $I(\emptyset) = (1)$ , since 1 is homogeneous of degree 0.

Every proper homogeneous ideal lies in  $R_+ = (x_0, \dots, x_n)$  (no nonzero constants), and  $\overline{Z}_K(R_+) = \emptyset = \overline{Z}_K((1))$ :  $R_+$  is the **irrelevant ideal**.

## Theorem (Projective Nullstellensatz)

Let  $K = \overline{K}$ . Then  $\overline{Z}_K$  and  $I$  are mutually inverse, inclusion-reversing bijections

$$\{\text{homogeneous radical ideals } I \subseteq R \text{ with } I \neq R_+\} \longleftrightarrow \{\text{subvarieties } X \subseteq \mathbb{P}_K^n\}.$$

For homogeneous  $I$ :  $\overline{Z}_K(I) = \emptyset$  iff  $\sqrt{I} \supseteq R_+$  iff  $R_d \subseteq I$  for some  $d$ ;  
and if  $\overline{Z}_K(I), \overline{Z}_K(J) \neq \emptyset$  then  $\overline{Z}_K(I) = \overline{Z}_K(J)$  iff  $\sqrt{I} = \sqrt{J}$ .

**Proof:** Apply the Nullstellensatz to the cone  $Z_K(I) \subseteq K^{n+1}$ ; see notes for details.

# The projective spectrum

## Definition

Let  $R$  be an  $\mathbb{N}$ -graded ring. Its **projective spectrum**  $\text{Proj}(R)$  is the set of homogeneous prime ideals  $\mathfrak{p} \not\supseteq R_+$ , partially ordered by inclusion, with the topology whose closed sets are  $V(I) := \{\mathfrak{p} \in \text{Proj}(R) : \mathfrak{p} \supseteq I\}$  for homogeneous ideals  $I \subseteq R$ .

## Example ( $K = \overline{K}$ , $R = K[x_0, \dots, x_n]$ )

For  $[a] \in \mathbb{P}_K^n$ ,  $\mathfrak{p}_a := I(\{[a]\})$  is generated by the linear forms vanishing at  $a$ .

It is prime ( $R/\mathfrak{p}_a \simeq K[t]$  via  $x_i \mapsto a_i t$ ) and  $\mathfrak{p}_a \not\supseteq R_+$ , so  $\mathfrak{p}_a \in \text{Proj}(R)$ .

The  $\mathfrak{p}_a$  are the maximal elements of  $\text{Proj}(R)$ , i.e. its closed points ( $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ ).

So  $\mathbb{P}_K^n$  is the set of closed points of  $\text{Proj}(R)$ , just as  $K^n = \text{Max}(K[x_1, \dots, x_n])$ .

A degree-preserving  $\varphi: R \rightarrow S$  induces  $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$  only on the open set of  $\mathfrak{p} \in \text{Proj}(S)$  with  $\varphi^{-1}(\mathfrak{p}) \not\supseteq R_+$ : e.g. for  $x, y \mapsto u, 0$  and  $\mathfrak{p} = (u) \subseteq K[u, v]$  we get  $\varphi^{-1}(\mathfrak{p}) = (x, y) = R_+$ . Maps of projective varieties are subtler; see 18.725.

# Localization

We now turn to the relationship between an ideal and its generating sets:  
when does an ideal have a minimal generating set, and to what extent is it unique?  
Positive answers: for homogeneous ideals in graded rings, and “locally”.  
Localization is the algebraic way to work locally: we make chosen elements invertible.

## Definition

A **multiplicative set** is a subset  $W \subseteq R$  with  $1 \in W$  and  $v, w \in W \Rightarrow vw \in W$ .

Examples:

- $W = \{1, f, f^2, f^3, \dots\}$  for any  $f \in R$ ;
- $W = R \setminus \mathfrak{p}$  for a prime ideal  $\mathfrak{p}$  (this is what it means for  $\mathfrak{p}$  to be prime!);
- the set of nonzerodivisors of  $R$ ;
- $R \setminus \bigcup_{\lambda} \mathfrak{p}_{\lambda}$  for any family of primes  $\mathfrak{p}_{\lambda}$  (intersections of multiplicative sets).

# Localization of a ring

## Definition

Let  $W \subseteq R$  be a multiplicative set. The **localization** of  $R$  at  $W$  is  $W^{-1}R := \{r/w : r \in R, w \in W\} / \sim$ , where  $r/w \sim r'/w'$  iff  $u(rw' - r'w) = 0$  for some  $u \in W$ ; operations:  $r/v + s/w := (rw + sv)/vw$ ,  $(r/v)(s/w) := rs/vw$ .

Check that  $\sim$  is an equivalence relation and that the operations are well defined! Why the  $u$ : if  $ur = 0$  with  $u \in W$  then  $r/1 = ur/u = 0/u = 0$  must hold.

The map  $\varphi: R \rightarrow W^{-1}R$ ,  $r \mapsto r/1$ , is a ring homomorphism, and each  $w/1$  is a unit with inverse  $1/w$ . Its kernel is  $\{r : wr = 0 \text{ for some } w \in W\}$ , so  $\varphi$  is injective iff  $W$  contains no zerodivisors, and  $W^{-1}R = 0$  iff  $0 \in W$ .

## Proposition (universal property)

If  $\alpha: R \rightarrow S$  is a ring map with  $\alpha(W) \subseteq S^\times$ , then  $\alpha = \beta \circ \varphi$  for a unique ring map  $\beta: W^{-1}R \rightarrow S$ , namely  $\beta(r/w) = \alpha(r)\alpha(w)^{-1}$  (well defined, check!).

# The most important localizations

Localizations we will use frequently:

- $R_f := W^{-1}R$  for  $W = \{1, f, f^2, \dots\}$ ; note that  $R_f \simeq R[x]/(fx - 1)$  (check!).  
E.g.  $K[x]_x = K[x, 1/x]$ ,  $\mathbb{Z}_f = \mathbb{Z}[1/f]$  for  $f \neq 0$ ,  $\mathbb{Z}_0 = 0$ , and  $(\mathbb{Z}/6\mathbb{Z})_2 \simeq \mathbb{Z}/3\mathbb{Z}$ .
- $R_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}R$  for a prime  $\mathfrak{p}$ , the **localization at  $\mathfrak{p}$** .  
E.g.  $\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} : p \nmid b\}$ , and  $K[x]_{(x)} = \{f/g \in K(x) : g(0) \neq 0\}$ .
- $W^{-1}R$  for  $W$  the set of nonzerodivisors: the **total ring of fractions** of  $R$ .  
Here  $R \rightarrow W^{-1}R$  is injective; if  $R$  is a domain this is its fraction field  $\text{Frac}(R)$ .

Geometric meaning ( $K = \overline{K}$ ,  $R = K[X]$  the coordinate ring of a subvariety  $X$ ):

- $R_f$  is the ring of functions  $g/f^n$  on the open set  $D(f) := \{a \in X : f(a) \neq 0\}$ ;
- $R_{\mathfrak{m}_a}$  is the ring of functions  $g/h$  with  $h(a) \neq 0$ , the **germs** of functions at  $a$ ;
- $\text{Frac}(R)$  (when  $R$  is a domain) is the field of rational functions on  $X$ .

# Localization of modules

## Definition

Let  $M$  be an  $R$ -module and  $W \subseteq R$  a multiplicative set. The **localization** of  $M$  at  $W$  is  $W^{-1}M := \{m/w : m \in M, w \in W\} / \sim$ , with  $m/w \sim m'/w'$  iff  $u(mw' - m'w) = 0$  for some  $u \in W$ . It is a  $W^{-1}R$ -module (formulas as for rings).

An  $R$ -linear map  $\alpha: M \rightarrow N$  induces a  $W^{-1}R$ -linear map  $W^{-1}\alpha: W^{-1}M \rightarrow W^{-1}N$ .

## Definition

The  **$W$ -torsion submodule** of  $M$  is  $M^{W\text{-tors}} := \{m \in M : wm = 0 \text{ for some } w \in W\}$ . For  $M = R$  this is an ideal, the kernel of  $R \rightarrow W^{-1}R$ .

## Lemma

$m/w = 0 \text{ in } W^{-1}M \iff m \in M^{W\text{-tors}} \iff \text{ann}_R(m) \cap W \neq \emptyset$ .

**Proof:**  $m/w \sim 0/1$  iff  $v(m \cdot 1 - 0 \cdot w) = vm = 0$  for some  $v \in W$ .

# Submodules, quotients, and the structure of $W^{-1}M$

## Lemma

For  $N \subseteq M$   $R$ -modules,  $W^{-1}N \subseteq W^{-1}M$  and  $W^{-1}(M/N) \simeq (W^{-1}M)/(W^{-1}N)$ .

**Proof:**  $n/w = 0$  in  $W^{-1}N$  iff  $n/w = 0$  in  $W^{-1}M$ , so  $W^{-1}N \rightarrow W^{-1}M$  is injective. The map  $W^{-1}M \rightarrow W^{-1}(M/N)$ ,  $m/w \mapsto \bar{m}/w$ , is surjective, and  $\bar{m}/w = 0$  iff  $vm \in N$  for some  $v \in W$ , i.e. iff  $m/w = vm/vw \in W^{-1}N$ .

So localization preserves injections and surjections (it is **exact**; more on this later).

## Proposition

Let  $\overline{M} := M/M^{W\text{-tors}}$ . Then  $W^{-1}M \simeq W^{-1}\overline{M} = \bigcup_{w \in W} \frac{1}{w}\overline{M}$ , and each  $m \mapsto m/w$  is an injection  $\overline{M} \hookrightarrow W^{-1}\overline{M}$  with image  $\frac{1}{w}\overline{M}$ .

**Proof:**  $W^{-1}(M^{W\text{-tors}}) = 0$  (torsion lemma), so  $W^{-1}M \simeq W^{-1}\overline{M}$  (lemma above). And  $\overline{M}$  is  $W$ -torsion-free (check!), so  $m/w = 0$  only if  $m = 0$  (torsion lemma again).

So  $W^{-1}M$  is a union of copies of  $\overline{M}$ , one for each denominator  $w \in W$ .

## Ideals in localizations

Let  $\varphi: R \rightarrow W^{-1}R$  be the localization map. For ideals  $I \subseteq R$  and  $J \subseteq W^{-1}R$  put  $IW^{-1}R := (\varphi(I))$ , the **expansion** of  $I$ , and  $J \cap R := \varphi^{-1}(J)$ , the **contraction** of  $J$ .

### Definition

The **saturation** of an ideal  $I \subseteq R$  is  $I^{W\text{-sat}} := \{r \in R : wr \in I \text{ for some } w \in W\}$ , an ideal containing  $I$ . We say  $I$  is  **$W$ -saturated** if  $I^{W\text{-sat}} = I$ .

### Lemma (expansion and contraction)

For  $I \subseteq R$  and  $J \subseteq W^{-1}R$  ideals,  $IW^{-1}R \cap R = I^{W\text{-sat}}$  and  $(J \cap R)W^{-1}R = J$ .

**Proof:** If  $r/1 = a/w$  with  $a \in I$ , then  $v(a - wr) = 0$  for some  $v \in W$ , so  $vwr = va \in I$  and  $r \in I^{W\text{-sat}}$ ; conversely, if  $wr = a \in I$  then  $r/1 = a/w \in IW^{-1}R$ . For the second,  $\subseteq$  is clear, and if  $b/w \in J$  then  $b/1 = (w/1)(b/w) \in J$ , so  $b \in J \cap R$  and  $b/w = (1/w)(b/1) \in (J \cap R)W^{-1}R$ .

In particular,  $IW^{-1}R = (1)$  iff  $1 \in I^{W\text{-sat}}$  iff  $I \cap W \neq \emptyset$ .

# Primes in localizations

## Proposition

$I \mapsto IW^{-1}R$  and  $J \mapsto J \cap R$  are inverse, inclusion-preserving bijections

$$\begin{aligned}\{W\text{-saturated ideals of } R\} &\longleftrightarrow \{\text{ideals of } W^{-1}R\}, \\ \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \cap W = \emptyset\} &\longleftrightarrow \operatorname{Spec}(W^{-1}R).\end{aligned}$$

Thus  $\varphi^*$  identifies  $\operatorname{Spec}(W^{-1}R)$  with the subspace  $\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \cap W = \emptyset\}$ .

**Proof:** To the blackboard!

## Example

- $\operatorname{Spec}(R_f) \simeq D(f) = \{\mathfrak{p} : f \notin \mathfrak{p}\}$ , an open subset of  $\operatorname{Spec}(R)$ .
- $\operatorname{Spec}(R_{\mathfrak{p}}) \simeq \{\mathfrak{q} : \mathfrak{q} \subseteq \mathfrak{p}\}$ , a “downward” analogue of  $V(\mathfrak{p}) = \{\mathfrak{q} : \mathfrak{q} \supseteq \mathfrak{p}\}$ ; in particular,  $\mathfrak{p}R_{\mathfrak{p}}$  is the unique maximal ideal of  $R_{\mathfrak{p}}$  (a **local ring**).
- For  $K = \overline{K}$  and  $R = K[X]$ : primes  $\mathfrak{p} \subseteq \mathfrak{m}_a$  correspond to subvarieties  $Z_K(\mathfrak{p}) \ni a$ , so  $\operatorname{Spec}(R_{\mathfrak{m}_a})$  only sees the geometry of  $X$  near  $a$ .

## On the blackboard: primes in localizations

**Proposition.**  $I \mapsto IW^{-1}R$  and  $J \mapsto J \cap R$  are mutually inverse, inclusion-preserving bijections between the  $W$ -saturated ideals of  $R$  and the ideals of  $W^{-1}R$ , and between the primes  $\mathfrak{p}$  of  $R$  with  $\mathfrak{p} \cap W = \emptyset$  and the primes of  $W^{-1}R$ . Thus  $\varphi^*: \text{Spec}(W^{-1}R) \rightarrow \text{Spec}(R)$  is a homeomorphism onto the subspace  $\{\mathfrak{p} : \mathfrak{p} \cap W = \emptyset\}$ .

**Proof.** By the lemma,  $(J \cap R)W^{-1}R = J$  for every ideal  $J$  of  $W^{-1}R$ , and  $IW^{-1}R \cap R = I^{W\text{-sat}} = I$  for every  $W$ -saturated ideal  $I$  of  $R$ . Moreover  $J \cap R$  is always  $W$ -saturated: if  $wr \in J \cap R$  with  $w \in W$  then  $r/1 = (1/w)(wr/1) \in J$ , so  $r \in J \cap R$ . Hence the two maps are mutually inverse bijections between  $W$ -saturated ideals of  $R$  and ideals of  $W^{-1}R$ , and both clearly preserve inclusions.

Let  $\mathfrak{p}$  be a prime of  $R$  with  $\mathfrak{p} \cap W = \emptyset$ . Then  $\mathfrak{p}$  is  $W$ -saturated: if  $wr \in \mathfrak{p}$  with  $w \in W$  then  $r \in \mathfrak{p}$ , since  $w \notin \mathfrak{p}$ . And  $\mathfrak{p}W^{-1}R$  is prime: it is proper because  $\mathfrak{p} \cap W = \emptyset$ , and if  $(a/v)(b/w) = p/u$  with  $p \in \mathfrak{p}$  then  $x(ua b - vwp) = 0$  for some  $x \in W$ , so  $xua b \in \mathfrak{p}$ ; as  $x, u \notin \mathfrak{p}$  this gives  $ab \in \mathfrak{p}$ , so  $a \in \mathfrak{p}$  or  $b \in \mathfrak{p}$ , i.e.  $a/v \in \mathfrak{p}W^{-1}R$  or  $b/w \in \mathfrak{p}W^{-1}R$ . Conversely, if  $\mathfrak{q}$  is a prime of  $W^{-1}R$  then  $\mathfrak{q} \cap R = \varphi^{-1}(\mathfrak{q})$  is prime, and it is disjoint from  $W$  because  $\varphi(W)$  consists of units. So the bijection restricts to primes, and on primes  $J \mapsto J \cap R$  is  $\varphi^*$ .

Finally,  $\varphi^*$  is continuous and injective, and it maps closed sets to closed subsets of its image:  $\varphi^*(V(J)) = V(J \cap R) \cap \text{im } \varphi^*$ , because for  $\mathfrak{q} \in \text{Spec}(W^{-1}R)$  we have  $\mathfrak{q} \supseteq J$  iff  $\mathfrak{q} \cap R \supseteq J \cap R$  (expand back using the bijection). So  $\varphi^*$  is a homeomorphism onto its image.  $\square$

*Presented on the blackboard; not part of the projected slides.*

# Fibers of maps on spectra

## Definition

Let  $\varphi: R \rightarrow S$  be a ring map and  $\mathfrak{p} \in \operatorname{Spec}(R)$ . The **fiber ring** of  $\varphi$  over  $\mathfrak{p}$  is  $\kappa_\varphi(\mathfrak{p}) := \varphi(R \setminus \mathfrak{p})^{-1}(S/\varphi(\mathfrak{p})S)$ , obtained by killing  $\varphi(\mathfrak{p})$  and inverting  $\varphi(R \setminus \mathfrak{p})$ .

## Proposition

$S \rightarrow \kappa_\varphi(\mathfrak{p})$  induces a homeomorphism from  $\operatorname{Spec}(\kappa_\varphi(\mathfrak{p}))$  onto the fiber  $(\varphi^*)^{-1}(\mathfrak{p})$ .

**Proof:**  $\operatorname{Spec}(S/\varphi(\mathfrak{p})S) \rightarrow \operatorname{Spec}(S)$  is a homeomorphism onto  $\{\mathfrak{q} : \varphi^{-1}(\mathfrak{q}) \supseteq \mathfrak{p}\}$ , and localizing at  $\varphi(R \setminus \mathfrak{p})$  then picks out the  $\mathfrak{q}$  with  $\varphi^{-1}(\mathfrak{q}) \subseteq \mathfrak{p}$ .

## Example

Let  $S := K[x, y, u, v]/(xu - yv)$ ,  $\varphi: R := K[x, y] \rightarrow S$  the inclusion, and  $\mathfrak{p} := (x)$ . Then  $S/xS = K[y, u, v]/(yv)$ , in which the image of  $R \setminus \mathfrak{p}$  is  $K[y] \setminus \{0\}$ ; inverting it kills  $v$  (as  $yv = 0$ ), so  $\kappa_\varphi(\mathfrak{p}) \simeq K(y)[u]$ . Thus the primes of  $S$  over  $(x)$  are  $(x, v)$  and  $(x, v, h)$  with  $h \in K[y, u]$  irreducible and  $h \notin K[y]$ .

# The “V-satz” and closed subsets of $\text{Spec}(R)$

## Proposition

Let  $I$  be an ideal of  $R$  and  $f \in R$ . Then  $f \in \mathfrak{p}$  for all  $\mathfrak{p} \in V(I)$  iff  $f \in \sqrt{I}$ ; that is,  $\bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \sqrt{I}$ .

In particular, the *nilradical*  $\sqrt{(0)}$  of  $R$  is the intersection of all its primes.

**Proof:**  $f \notin \sqrt{I}$  iff  $I \cap \{1, f, f^2, \dots\} = \emptyset$  iff  $IR_f \neq (1)$  iff  $IR_f \subseteq \mathfrak{q}$  for some  $\mathfrak{q} \in \text{Spec}(R_f)$  iff  $I \subseteq \mathfrak{p}$  for some prime  $\mathfrak{p}$  of  $R$  with  $f \notin \mathfrak{p}$  (as  $\text{Spec}(R_f) \simeq D(f)$ ).

$R_f \simeq R[y]/(fy - 1)$ : this is the Rabinowitsch trick from the Strong Nullstellensatz!

## Corollary

$I \mapsto V(I)$  and  $Y \mapsto \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$  are inverse, inclusion-reversing bijections between radical ideals of  $R$  and closed subsets of  $\text{Spec}(R)$ ; so  $V(I) = V(J)$  iff  $\sqrt{I} = \sqrt{J}$ .

**Proof:** As for subvarieties of  $K^d$ :  $\bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \sqrt{I}$  by the proposition, and  $V(\bigcap_{\mathfrak{p} \in V(J)} \mathfrak{p}) = V(\sqrt{J}) = V(J)$ .