

18.705 Commutative Algebra

Lecture 2

Adapted from the [2018 lecture notes](#) of Jack Jeffries (§1.5, §2.1, §2.2)

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Graded rings

Definition

Let A be a commutative monoid (e.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N}^n$). A ring R is **A -graded** if $R = \bigoplus_{a \in A} R_a$ (as abelian groups) with $1 \in R_0$ and $R_a R_b \subseteq R_{a+b}$ for all a, b . Elements of R_a are **homogeneous** of **degree** a ; for $0 \neq r \in R_a$ we write $|r| = a$.

($1 \in R_0$ is automatic when A is cancellative, e.g. $A = \mathbb{N}, \mathbb{Z}, \mathbb{N}^n$; check!)

Every $r \in R$ is uniquely $r = \sum_{a \in A} r_a$ with $r_a \in R_a$, almost all zero;
the r_a are the **homogeneous components** of r .

Examples with $R = K[x_1, \dots, x_n]$ and K a field:

- the **standard grading**: R_d is the K -span of the monomials of total degree d ;
- the **grading with weights** $(\beta_1, \dots, \beta_n) \in \mathbb{N}^n$: $|x_i| = \beta_i$, so $|x_1^{\alpha_1} \cdots x_n^{\alpha_n}| = \sum_i \alpha_i \beta_i$;
e.g. $x_1^2 + x_2^3$ is homogeneous of degree 6 for the weights $(3, 2)$;
- the **fine grading** by $\mathbb{N}^n$: $R_{(d_1, \dots, d_n)} = Kx_1^{d_1} \cdots x_n^{d_n}$.

Graded subrings and quotients

A ring map $\varphi: R \rightarrow S$ of A -graded rings is **degree-preserving** if $\varphi(R_a) \subseteq S_a$.

- For a submonoid $\Gamma \subseteq \mathbb{N}^n$, $K[\Gamma] := \bigoplus_{\gamma \in \Gamma} Kx^\gamma$ is an $\mathbb{N}^n$ -graded subring of $K[x]$; every $\mathbb{N}^n$ -graded K -subalgebra of $K[x] = K[x_1, \dots, x_n]$ has this form (check!).
- If G acts on R by degree-preserving automorphisms then R^G is a graded subring (the homogeneous components of an invariant element are invariant).

Definition

An ideal $I \subseteq R$ is **homogeneous** if it is generated by homogeneous elements.

Equivalently, $f \in I$ iff every homogeneous component of f lies in I (check!).
Then $R/I = \bigoplus_a R_a / (I \cap R_a)$ is graded, and $R \rightarrow R/I$ is degree-preserving.

Example: $K[x, y, z]/(x^2 + y^3 + z^5)$ is not graded with $|x| = |y| = |z| = 1$, but it is with $(|x|, |y|, |z|) = (15, 10, 6)$.

Noetherian graded rings

Let $R = \bigoplus_{d \geq 0} R_d$ be an $\mathbb{N}$ -graded ring. Then R_0 is a subring of R , each R_d is an R_0 -module, and $R_+ := \bigoplus_{d > 0} R_d$ is a homogeneous ideal with $R/R_+ \simeq R_0$.

Proposition

Let R be an $\mathbb{N}$ -graded ring and let $f_1, \dots, f_n \in R$ be homogeneous of positive degree. Then $R_+ = (f_1, \dots, f_n)$ if and only if $R = R_0[f_1, \dots, f_n]$, and R is Noetherian if and only if R_0 is Noetherian and R is algebra-finite over R_0 .

Proof: To the blackboard!

On the blackboard: Noetherian graded rings

Proposition. Let R be an $\mathbb{N}$ -graded ring and $f_1, \dots, f_n \in R$ homogeneous of positive degree. Then $R_+ = (f_1, \dots, f_n)$ if and only if $R = R_0[f_1, \dots, f_n]$. It follows that R is Noetherian if and only if R_0 is Noetherian and R is algebra-finite over R_0 .

Proof. ($\Leftarrow$) Let $r \in R_+$ and write $r = P(f_1, \dots, f_n)$ with $P \in R_0[x_1, \dots, x_n]$. Each monomial of P of positive degree evaluates to an element of R_+ (since $|f_i| > 0$), so comparing degree-0 components shows that the constant term of P is $r_0 = 0$. Every other monomial of P is divisible by some x_i , so $r \in (f_1, \dots, f_n)$.

($\Rightarrow$) Since R is spanned by homogeneous elements it suffices to show $R_d \subseteq R_0[f_1, \dots, f_n]$ for all d , by induction on d ; the case $d = 0$ is trivial. For $r \in R_d$ with $d > 0$ we have $r \in R_+ = (f_1, \dots, f_n)$, so $r = \sum_i \alpha_i f_i$ with $\alpha_i \in R$, and taking degree- d components we may assume $\alpha_i \in R_{d-|f_i|}$ (with $\alpha_i = 0$ if $d < |f_i|$). By the inductive hypothesis each α_i lies in $R_0[f_1, \dots, f_n]$, hence so does r .

If R is Noetherian then $R_0 \simeq R/R_+$ is Noetherian and $R_+ = (g_1, \dots, g_m)$ for some $g_j \in R_+$; replacing each g_j by its homogeneous components (which lie in R_+ , since R_+ is homogeneous) we may assume the g_j are homogeneous of positive degree, and then $R = R_0[g_1, \dots, g_m]$ by the first part. Conversely, if R_0 is Noetherian and R is algebra-finite over R_0 then R is Noetherian by the Hilbert Basis Theorem. $\square$

Presented on the blackboard; not part of the projected slides.

Systems of polynomial equations

Question

To what extent is a system of polynomial equations determined by its solution set?

At first glance, things look bad: the systems

$$\{x = 0, y = 0\}, \quad \{x^2 = -y^2\}, \quad \{xy^2 + x = -x^3, yx^2 + y = -y^3\}$$

all have the solution set $\{(0, 0)\}$ in $\mathbb{R}^2$, and the systems

$$\{2x + 3y + 4z + 5w = 1, 4x + 6y + 8z + 10w = 3\}, \quad \{x^2 + y^2 + z^2 + w^2 = 7, xyzw = 4\}$$

have empty solution sets in $\mathbb{Q}^4$.

Over $\mathbb{R}$ or $\mathbb{Q}$, many polynomials in one variable have no roots at all.

But over $K = \overline{K}$, a nonzero $f \in K[z]$ is determined up to a scalar by its roots, provided we take multiplicities into account. This suggests restricting to $K = \overline{K}$.

Zero sets and affine varieties

Any system of polynomial equations in d variables over a field K can be written as $\{f_\lambda(x) = 0 : \lambda \in \Lambda\}$ with $f_\lambda \in K[x] := K[x_1, \dots, x_d]$. Its **zero set** in K^d is

$$Z_K(\{f_\lambda\}) := \{a \in K^d : f_\lambda(a) = 0 \text{ for all } \lambda \in \Lambda\}.$$

Every element of the ideal $I := (f_\lambda : \lambda \in \Lambda)$ is a consequence of the system, since $\sum_i r_i f_{\lambda_i}$ vanishes wherever all the f_{λ_i} do. Thus $Z_K(\{f_\lambda\}) = Z_K(I)$. By the Hilbert Basis Theorem, $I = (f_1, \dots, f_t)$ for some $f_i \in I$: every system of polynomial equations is equivalent to a finite one!

Definition

An **affine variety** in K^d is a subset of the form $Z_K(I)$ for some ideal $I \subseteq K[x]$.

We have $\{f_\lambda\} \mapsto I \mapsto Z_K(I)$; our question is when $I \mapsto Z_K(I)$ can be reversed. Note that $I \subseteq J$ implies $Z_K(I) \supseteq Z_K(J)$ (more equations, fewer solutions).

Examples of affine varieties

Some examples:

- Linear subspaces of K^d are affine varieties.
- Finite unions of affine varieties are affine varieties: $Z_K(I) \cup Z_K(J) = Z_K(IJ)$;
e.g. $Z_K(xy)$ is the union of the two coordinate axes in K^2 . (Check the equality!)
- The union X of the planes $\{x = y = 0\}$ and $\{u = v = 0\}$ in K^4 is $Z_K(xu, xv, yu, yv)$, and also $Z_K(xu, xv + yu, yv)$: a point of the latter not in X would have $xv = -yu \neq 0$, hence $xu \neq 0$, a contradiction.
So different ideals can have the same zero set; note $xv \notin (xu, xv + yu, yv)$.
- Let I be the ideal generated by the 2×2 minors of $\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$.
Then $Z_K(I)$ is the set of 2×3 matrices of rank at most one.

Solutions as algebra homomorphisms

Let R be a ring. A **homomorphism of R -algebras** $S \rightarrow T$ is a ring homomorphism compatible with the structure maps $R \rightarrow S$ and $R \rightarrow T$;
let $\text{Hom}_{R\text{-alg}}(S, T)$ denote the set of such maps.

Let S be an R -algebra, let $\Lambda \subseteq R[x] := R[x_1, \dots, x_d]$, and let $I := (\Lambda)$.
The solutions in S^d of the system $\{f(x) = 0 : f \in \Lambda\}$ form the set
 $Z_S(I) := \{a \in S^d : f(a) = 0 \text{ for all } f \in I\}$.

Proposition

There is a bijection $Z_S(I) \rightarrow \text{Hom}_{R\text{-alg}}(R[x]/I, S)$ that sends $a \in Z_S(I)$ to the R -algebra map $x_i \mapsto a_i$.

Proof: The map $R[x] \rightarrow S$, $x_i \mapsto a_i$, kills Λ , hence I , so it factors through $R[x]/I$.
Conversely, an R -algebra map $R[x]/I \rightarrow S$ is determined by the images a_i of the x_i , and $f(a) = 0$ for all $f \in I$.

Points and maximal ideals

Let K be a field and $S := K[x_1, \dots, x_d]$. For $a \in K^d$ let $\mathfrak{m}_a := (x_1 - a_1, \dots, x_d - a_d)$, the kernel of $f \mapsto f(a)$; it is a maximal ideal of S with $S/\mathfrak{m}_a = K$.

Theorem

Let $R := S/I$ for an ideal $I \subseteq S$. There are bijections

$$Z_K(I) \longleftrightarrow \{ \text{maximal ideals } \mathfrak{m} \text{ of } R \text{ with } R/\mathfrak{m} = K \} \longleftrightarrow \operatorname{Hom}_{K\text{-alg}}(R, K)$$

given by $a \mapsto \mathfrak{m}_a/I$ and $\mathfrak{m} \mapsto (R \twoheadrightarrow R/\mathfrak{m} = K)$;

here $R/\mathfrak{m} = K$ means that $K \rightarrow R \rightarrow R/\mathfrak{m}$ is an isomorphism.

Proof: To the blackboard!

For $I = 0$: the maximal ideals $\mathfrak{m}$ of S with $S/\mathfrak{m} = K$ are exactly the $\mathfrak{m}_a$ with $a \in K^d$. A maximal ideal $\mathfrak{m}$ may have $R/\mathfrak{m} \neq K$, e.g. $(x^2 + 1) \subseteq \mathbb{R}[x]$; but if $K = \overline{K} \dots$

On the blackboard: points and maximal ideals

Theorem. Let K be a field, $S := K[x_1, \dots, x_d]$, $I \subseteq S$ an ideal, and $R := S/I$. Then $a \mapsto \mathfrak{m}_a/I$ and $\mathfrak{m} \mapsto (R \twoheadrightarrow R/\mathfrak{m} = K)$ are bijections

$$Z_K(I) \longleftrightarrow \{\text{maximal ideals } \mathfrak{m} \text{ of } R \text{ with } R/\mathfrak{m} = K\} \longleftrightarrow \text{Hom}_{K\text{-alg}}(R, K).$$

Proof. First take $I = 0$. For $a \in K^d$, $\mathfrak{m}_a$ is the kernel of the surjection $S \rightarrow K$, $f \mapsto f(a)$, so $\mathfrak{m}_a$ is maximal with $S/\mathfrak{m}_a = K$. If $\mathfrak{m}_a = \mathfrak{m}_b$ then $a_i - b_i = (x_i - b_i) - (x_i - a_i) \in \mathfrak{m}_a$ for each i ; a proper ideal contains no nonzero constants, so $a = b$. Now let $\mathfrak{m}$ be a maximal ideal of S with $S/\mathfrak{m} = K$, let $\pi: S \rightarrow K$ be the quotient map, and put $a_i := \pi(x_i)$. Then $\pi(x_i - a_i) = 0$, so $\mathfrak{m}_a \subseteq \mathfrak{m}$, and $\mathfrak{m} = \mathfrak{m}_a$ because $\mathfrak{m}_a$ is maximal. Thus $a \mapsto \mathfrak{m}_a$ is a bijection from K^d to the set of maximal ideals $\mathfrak{m}$ of S with $S/\mathfrak{m} = K$.

For general I , the maximal ideals of $R = S/I$ are $\mathfrak{m}/I$ with $\mathfrak{m}$ a maximal ideal of S containing I , and $R/(\mathfrak{m}/I) \simeq S/\mathfrak{m}$. Since $\mathfrak{m}_a \supseteq I$ if and only if $f(a) = 0$ for $f \in I$ ($a \in Z_K(I)$), the first bijection follows.

For the second, a K -algebra map $\varphi: R \rightarrow K$ is surjective (it is the identity on K), so its kernel $\mathfrak{m}$ is a maximal ideal with $R/\mathfrak{m} = K$, and φ is the composition $R \twoheadrightarrow R/\mathfrak{m} = K$. Conversely, $R \twoheadrightarrow R/\mathfrak{m} = K$ is a K -algebra map with kernel $\mathfrak{m}$. □

Presented on the blackboard; not part of the projected slides.

Zariski's Lemma

Lemma (Zariski)

Let $K \subseteq L$ be fields. If L is a finitely generated K -algebra then $[L : K] < \infty$.

Proof: To the blackboard!

Corollary (Weak Nullstellensatz)

If R is a finitely generated K -algebra and $\mathfrak{m} \in \text{Max}(R)$, then $[R/\mathfrak{m} : K] < \infty$.

In particular, if $K = \overline{K}$ then $R/\mathfrak{m} = K$, so $a \mapsto \mathfrak{m}_a$ and $a \mapsto \mathfrak{m}_a/I$ give bijections $K^d \rightarrow \text{Max}(K[x_1, \dots, x_d])$ and $Z_K(I) \rightarrow \text{Max}(K[x_1, \dots, x_d]/I)$.

Proof: $R/\mathfrak{m}$ is a field and a finitely generated K -algebra, so Zariski applies; for the bijections, use the earlier theorem on points and maximal ideals.

Exercise: residue fields $R/\mathfrak{m}$ of finitely generated $\mathbb{Z}$ -algebras R are finite.

On the blackboard: proof of Zariski's Lemma

Lemma (Zariski). Let $K \subseteq L$ be fields with $L = K[a_1, \dots, a_d]$. Then $[L : K] < \infty$.

Proof. Reorder the a_i so that $a_1, \dots, a_c$ is a transcendence basis for L/K , let $K' := K(a_1, \dots, a_c) \subseteq L$. Then $L = K'[a_{c+1}, \dots, a_d]$ is algebra-finite over K' and generated by algebraic, hence integral, elements, so L is module-finite over K' . If $c = 0$ then $K' = K$ and we are done, so suppose $c \geq 1$.

Apply the Artin–Tate lemma to $K \subseteq K' \subseteq L$: K is Noetherian, L is module-finite over K' and algebra-finite over K , so K' is algebra-finite over K , say $K' = K[h_1, \dots, h_m]$. Now K' is the fraction field of the polynomial ring $P := K[a_1, \dots, a_c]$ (the a_i are algebraically independent), so we may write $h_i = f_i/g_i$ with $f_i, g_i \in P$, and then $K' \subseteq P[1/g]$, where $g := g_1 \cdots g_m$.

But P is a UFD with infinitely many pairwise non-associate primes (Euclid: given primes $p_1, \dots, p_r$, any prime factor of $p_1 \cdots p_r + 1$ is a new one), so there is a prime $p \in P$ with $p \nmid g$.

Since $1/p \in K' \subseteq P[1/g]$ we have $1/p = v/g^N$ for some $v \in P$ and $N \geq 0$, so $g^N = pv$ and $p \mid g$, a contradiction. Thus $c = 0$ and $[L : K] < \infty$. □

Remark. The case $c \geq 1$ shows that $K(x_1, \dots, x_c)$ is not a finitely generated K -algebra; this is really all the proof uses.

Presented on the blackboard; not part of the projected slides.

The Medium Nullstellensatz

Theorem (Medium Nullstellensatz)

Let $K = \overline{K}$ and $I \subseteq K[x_1, \dots, x_d]$ an ideal. Then $Z_K(I) = \emptyset \Leftrightarrow I = (1)$.

Proof: If $I \neq (1)$ then $I \subseteq \mathfrak{m}_a$ for some $a \in K^d$ (Zorn's lemma and the weak NSS), so $a \in Z_K(\mathfrak{m}_a) \subseteq Z_K(I)$.

Corollary

If a system of polynomial equations over a field K has a solution in some nonzero K -algebra, then it has a solution in $\overline{K}$.

Proof: Let $L := \overline{R/\mathfrak{m}}$ for some $\mathfrak{m} \in \text{Max}(R)$, and fix a K -embedding $\overline{K} \rightarrow L$. A solution in R gives one in L . If there is none in $\overline{K}$ then $1 = \sum_i g_i f_{\lambda_i}$ with $g_i \in \overline{K}[x]$ (Medium NSS), hence also in $L[x]$; so there is none in L either.

So as far as the existence of solutions (in some K -algebra) is concerned, nothing is lost by passing to $\overline{K}$; we will mostly take $K = \overline{K}$ from now on.