18.705 Commutative Algebra
Lecture 18

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Recap, and where we are going

Lecture 17: an ideal I of R defines the I-adic topology on every R-module M, and the I-adic completion $\widehat{M}_I := \{\text{Cauchy sequences}\}/\{\text{null sequences}\} \simeq \varprojlim_n M/I^nM$.

Today: for finitely generated modules over a Noetherian ring, completion is exact, and $\widehat{R}$ is flat and Noetherian. Then: the completion of a Noetherian local ring.

- $M \to \widehat{M}_I$ has kernel $\bigcap_n I^nM$; M is *I-adically complete* if it is an isomorphism.
- Artin–Rees: if R is Noetherian and $N \subseteq M$ are finitely generated R-modules, then there is a $c \geq 0$ such that $I^{n+c}M \cap N \subseteq I^nN$ for all $n \geq 0$.
- Krull: if $(R, \mathfrak{m})$ is Noetherian local and M is finitely generated, then $\bigcap_n \mathfrak{m}^nM = 0$. So $R \subseteq \widehat{R}$; for example, $K[x]_{(x)} \subseteq K[[x]]$ and $\mathbb{Z}_{(p)} \subseteq \mathbb{Z}_p$.
- $\mathrm{gr}_I(R) := \bigoplus_n I^n/I^{n+1}$. For $R = K[x,y]/(y^2 - x^2(x+1))$ and $\mathfrak{m} = (x,y)$ we have $\mathrm{gr}_{\mathfrak{m}}(R) \simeq K[x,y]/(y^2 - x^2)$, the coordinate ring of the node's tangent cone.
- Flatness (Lectures 7–8): $- \otimes_R N$ is always right exact, and N is flat iff $J \otimes_R N \to N$ is injective for every finitely generated ideal $J \subseteq R$ (Problem Set 3).

Completing finitely generated modules

Theorem

Let R be a Noetherian ring and I an ideal; write $\widehat{M}$ for the I-adic completion $\widehat{M}_I$.

1. *If $0 \to N \to M \to P \to 0$ is an exact sequence of finitely generated R-modules, then $0 \to \widehat{N} \to \widehat{M} \to \widehat{P} \to 0$ is exact.*
2. *For every finitely generated R-module M, the map $\widehat{R} \otimes_R M \to \widehat{M}$ defined by $[(r_i)] \otimes m \mapsto [(r_i m)]$ is an isomorphism.*
3. *$\widehat{R}$ is a flat R-algebra.*

Proof: To the blackboard!

- For every ideal J of R we get $\widehat{J} \simeq J\widehat{R}$ and $\widehat{R/J} \simeq \widehat{R}/J\widehat{R}$.
- For $J = I^n$ this gives $\widehat{R}/I^n\widehat{R} \simeq R/I^n$, since I is nilpotent on R/I^n. So $\widehat{R}$ is $I\widehat{R}$-adically complete, and $\mathrm{gr}_{I\widehat{R}}(\widehat{R}) \simeq \mathrm{gr}_I(R)$.
- Finite generation matters: for $R = \mathbb{Z}$, $I = (p)$, $M = \mathbb{Q}$ we have $\widehat{M} = 0$, since $p^n\mathbb{Q} = \mathbb{Q}$, but $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$, since $\mathbb{Z}_p$ is torsion free.

On the blackboard: completion is exact

Theorem (1). Let R be Noetherian and $0 \to N \to M \xrightarrow{\pi} P \to 0$ an exact sequence of finitely generated R-modules. Then $0 \to \widehat{N} \to \widehat{M} \to \widehat{P} \to 0$ is exact (completions are I-adic).

Proof. We may assume $N \subseteq M$. By Artin–Rees there is a c with $I^{n+c}M \cap N \subseteq I^n N$ for all n. Note that a subsequence of a Cauchy sequence is Cauchy, and equivalent to it (their difference is null).

$\widehat{N} \to \widehat{M}$ *is injective.* Let (x_i) be a Cauchy sequence in N that is a null sequence in M. For each n we have $x_i \in I^{n+c}M \cap N \subseteq I^n N$ for all large i. So (x_i) is a null sequence in N.

$\widehat{M} \to \widehat{P}$ *is surjective.* Let (p_i) be a Cauchy sequence in P. Passing to a subsequence, we may assume that $p_{i+1} - p_i \in I^i P = \pi(I^i M)$ for all i. Choose $m_1 \in M$ with $\pi(m_1) = p_1$ and $d_i \in I^i M$ with $\pi(d_i) = p_{i+1} - p_i$, and let $m_{i+1} := m_i + d_i$. Then $m_j - m_i \in I^i M$ for $j \geq i$, so (m_i) is Cauchy, and $\pi(m_i) = p_i$ for all i.

Exactness at $\widehat{M}$. The composition $\widehat{N} \to \widehat{M} \to \widehat{P}$ is zero. Let (m_i) be a Cauchy sequence in M such that $(\pi(m_i))$ is a null sequence in P. Passing to a subsequence, we may assume $\pi(m_i) \in I^i P = \pi(I^i M)$, so that $m_i = x_i + e_i$ with $x_i \in \ker \pi = N$ and $e_i \in I^i M$. Then (e_i) is a null sequence, so (x_i) is a Cauchy sequence in M equivalent to (m_i). For each n we have $x_i - x_j \in I^{n+c}M \cap N \subseteq I^n N$ for all large i, j, so (x_i) is a Cauchy sequence in N, and its class in $\widehat{N}$ maps to the class of (m_i) in $\widehat{M}$. □

Presented on the blackboard; not part of the projected slides.

On the blackboard: $\widehat{M} \simeq \widehat{R} \otimes_R M$, and $\widehat{R}$ is flat

Theorem (2). Let R be Noetherian and M a finitely generated R-module. Then $\theta_M : \widehat{R} \otimes_R M \to \widehat{M}$, $[(r_i)] \otimes m \mapsto [(r_i m)]$, is an isomorphism.

Proof. The map θ_M is well defined and natural in M. Clearly θ_R is an isomorphism, and so is θ_{R^n}, since both sides commute with finite direct sums. Choose an exact sequence $0 \to L \to R^n \to M \to 0$; then L is finitely generated, because R is Noetherian. In the commutative diagram

$$\begin{array}{ccccccccc} & & \widehat{R} \otimes_R L & \longrightarrow & \widehat{R} \otimes_R R^n & \longrightarrow & \widehat{R} \otimes_R M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_L} & & \simeq\downarrow{\scriptstyle \theta_{R^n}} & & \downarrow{\scriptstyle \theta_M} & & \\ 0 & \longrightarrow & \widehat{L} & \longrightarrow & \widehat{R^n} & \longrightarrow & \widehat{M} & \longrightarrow & 0 \end{array}$$

the top row is exact because tensor products are right exact, and the bottom row is exact by (1).

Surjective: $\widehat{R^n} \to \widehat{M}$ is surjective and θ_{R^n} is an isomorphism, so θ_M is surjective. This holds for every finitely generated module, so θ_L is surjective as well.

Injective: let $x \in \ker \theta_M$ and lift it to $y \in \widehat{R} \otimes_R R^n$. Then $\theta_{R^n}(y)$ maps to 0 in $\widehat{M}$, so it is the image of some $\theta_L(z)$ with $z \in \widehat{R} \otimes_R L$. Let y' be the image of z in $\widehat{R} \otimes_R R^n$. Then $\theta_{R^n}(y') = \theta_{R^n}(y)$, so $y' = y$, and x is the image of z under the composition of the maps in the top row, which is zero. □

Theorem (3). $\widehat{R}$ is flat over R. **Proof.** By Problem Set 3 it suffices to show that $\widehat{R} \otimes_R J \to \widehat{R}$ is injective for every ideal J. Under θ_J and θ_R this is the map $\widehat{J} \to \widehat{R}$, which is injective by (1). □

Corollary. $\widehat{J} \simeq \widehat{R} \otimes_R J \simeq J\widehat{R}$ (the last by flatness), and $\widehat{R/J} \simeq \widehat{R} \otimes_R R/J \simeq \widehat{R}/J\widehat{R}$.

Presented on the blackboard; not part of the projected slides.

The completion of a Noetherian ring

Proposition

If R is Noetherian then $\widehat{R}_I$ is Noetherian, for every ideal I of R.

Proof: Let $I = (a_1, \ldots, a_r)$. The homomorphism $R[[x_1, \ldots, x_r]] \to \widehat{R}_I$ with $x_i \mapsto a_i$ is surjective, since every element of $\widehat{R}_I$ is a sum $\sum_d g_d$ with $g_d \in I^d$ (blackboard). And $R[[x_1, \ldots, x_r]]$ is Noetherian, by the Hilbert basis theorem. □

Now let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and $\widehat{R}$ its completion.

- $\widehat{R}$ is a complete Noetherian local ring, with maximal ideal $\mathfrak{m}\widehat{R}$ and residue field k: if $a \in \mathfrak{m}\widehat{R}$ then $1 - a$ is a unit, with inverse $1 + a + a^2 + \cdots$.
- $R \subseteq \widehat{R}$ (Krull), $\widehat{R}$ is flat over R, and $J\widehat{R} \cap R = J$ for every ideal J of R.
- $\mathfrak{m}\widehat{R}/\mathfrak{m}^2\widehat{R} \simeq \mathfrak{m}/\mathfrak{m}^2$ and $\dim \widehat{R} = \dim R$ (exercise). So R is regular iff $\widehat{R}$ is.

On the blackboard: $\widehat{R}_I$ is a quotient of $R[[x_1, \dots, x_r]]$

Infinite sums. Let $g_d \in I^d$ for $d \geq 0$. The partial sums $s_n := \sum_{d<n} g_d$ form a Cauchy sequence in R, since $s_m - s_n \in I^n$ for $m \geq n$, and we *define* $\sum_d g_d \in \widehat{R}_I$ to be its class. Under $\widehat{R}_I \simeq \varprojlim_n R/I^n$ it is the element whose image in R/I^n is the image of the finite sum s_n, for every n.

Step 1. Every element of $\widehat{R}_I$ is $\sum_d g_d$ for some $g_d \in I^d$. **Proof.** Let $(r_i)_{i\geq 1}$ be a Cauchy sequence in R. Passing to a subsequence (which is equivalent to it), we may assume that $r_{i+1} - r_i \in I^i$ for all i. Let $g_0 := r_1$ and $g_d := r_{d+1} - r_d \in I^d$ for $d \geq 1$. Then $s_n = r_n$ for all $n \geq 1$, so $[(r_i)] = \sum_d g_d$. □

Step 2. Let $I = (a_1, \dots, a_r)$ and $a := (a_1, \dots, a_r)$. Write $F \in R[[x_1, \dots, x_r]]$ as $F = \sum_d F_d$ with F_d homogeneous of degree d, so that $F_d(a) \in I^d$. Then $\varphi(F) := \sum_d F_d(a)$ defines a ring homomorphism $\varphi\colon R[[x_1, \dots, x_r]] \to \widehat{R}_I$ with $\varphi(x_i) = a_i$.

Proof. $\varphi(F)$ is the class of $(F_{<n}(a))_n$, where $F_{<n} := \sum_{d<n} F_d$ is a polynomial. So $\varphi(1) = 1$, and φ is additive. Every term of $(FG)_{<n} - F_{<n}G_{<n}$ has degree at least n, so its value at a lies in I^n. Thus $((FG)_{<n}(a))_n$ and $(F_{<n}(a)G_{<n}(a))_n$ differ by a null sequence, and $\varphi(FG) = \varphi(F)\varphi(G)$. □

Step 3. φ is surjective, so $\widehat{R}_I \simeq R[[x_1, \dots, x_r]]/\ker\varphi$ is Noetherian if R is. **Proof.** Let $g_d \in I^d$ for $d \geq 0$. The ideal I^d is generated by the monomials of degree d in $a_1, \dots, a_r$, so $g_d = G_d(a)$ with G_d homogeneous of degree d. Then $G := \sum_d G_d$ is a power series and $\varphi(G) = \sum_d g_d$; apply Step 1. □

Remark. Let R be Noetherian. For $h_d \in I^d\widehat{R}_I$ the partial sums of $\sum_d h_d$ are a Cauchy sequence for the $I\widehat{R}_I$-adic topology on $\widehat{R}_I$. As $\widehat{R}_I$ is $I\widehat{R}_I$-adically complete, they have a unique limit in $\widehat{R}_I$, and $\sum_d h_d$ denotes this limit.

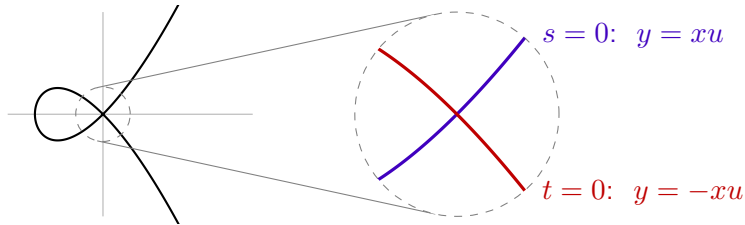
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Example: completing the node

Let $\operatorname{char} K \neq 2$, and let $R := S/(y^2 - x^2(x+1))$ with $S := K[x,y]_{(x,y)}$, the local ring of the node. It is a domain of dimension 1 that is not normal (Lecture 16).

- $\widehat{R} \simeq K[[x,y]]/(y^2 - x^2(x+1))$, because $\widehat{S/J} \simeq \widehat{S}/J\widehat{S}$.
- In $K[[x]]$ we have $1 + x = u^2$ with $u = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$. So in $K[[x,y]]$,
$$y^2 - x^2(x+1) = (y - xu)(y + xu).$$
- So $\widehat{R} \simeq K[[s,t]]/(st)$, with $s := y - xu$ and $t := y + xu$. It is not a domain!
- The completion sees the two branches of the curve through the node.
Compare $\operatorname{gr}_{\mathfrak{m}}(R) \simeq K[x,y]/(y^2 - x^2)$, which sees their tangent lines.
- For the cusp, $K[[x,y]]/(y^2 - x^3) \simeq K[[t^2, t^3]]$ is a domain: only one branch.

Example: the two branches of the node



- R is a domain: the curve is irreducible, and its two branches at the origin are joined by the loop. Localizing at the origin does not separate them.
- $\widehat{R} \simeq K[[s,t]]/(st)$ has two minimal primes, (s) and (t): one for each branch.
- For $K = \mathbb{R}$ the series $u = \sqrt{1+x}$ converges for $|x| < 1$, so the branches are honest curves $y = \pm xu$ near the origin; they meet again at $x = -1$.

The Cohen structure theorem

Theorem (Cohen)

Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring of dimension d.

1. *If R contains a field, then it contains a coefficient field (a subfield that maps isomorphically onto k), and $R \simeq k[[x_1, \ldots, x_n]]/J$.*
   Otherwise $R \simeq V[[x_1, \ldots, x_n]]/J$ with (V, pV, k) a complete DVR, $p = \operatorname{char} k$.
2. *R is regular iff $R \simeq k[[x_1, \ldots, x_d]]$, or $R \simeq V[[x_1, \ldots, x_{d-1}]]$, or $R \simeq V[[x_1, \ldots, x_d]]/(p - f)$ with $f \in (p, x_1, \ldots, x_d)^2$ (with V as in (1)).*
3. *If R contains a field or is a domain, then R is module-finite over a subring isomorphic to $k[[y_1, \ldots, y_d]]$ or $V[[y_1, \ldots, y_{d-1}]]$.*

- We will not prove this. Compare (1) and (3) with affine algebras $K[x_1, \ldots, x_n]/J$ and Noether normalization (Lecture 12).
- Strategy: localize, complete, and then work in a quotient of a power series ring.

Summary

- For finitely generated modules over a Noetherian ring R, completion is exact and $\widehat{M} \simeq \widehat{R} \otimes_R M$. The ring $\widehat{R}$ is Noetherian, and flat over R.
- $\widehat{R/J} \simeq \widehat{R}/J\widehat{R}$ for every ideal J, and $\widehat{R}/I^n\widehat{R} \simeq R/I^n$.
  So $\widehat{R}$ is $I\widehat{R}$-adically complete, and $\mathrm{gr}_{I\widehat{R}}(\widehat{R}) \simeq \mathrm{gr}_I(R)$.
- The completion of a Noetherian local ring $(R, \mathfrak{m}, k)$ is a complete Noetherian local ring $\widehat{R} \supseteq R$ with maximal ideal $\mathfrak{m}\widehat{R}$, residue field k, and $\dim \widehat{R} = \dim R$.
  It is regular iff R is regular.
- The local ring R of the node is a domain, but $\widehat{R} \simeq K[[s,t]]/(st)$ is not:
  the completion sees the two branches, and $\mathrm{gr}_{\mathfrak{m}}(R)$ sees their tangent lines.
- Cohen: complete Noetherian local rings are quotients of power series rings over a field or a complete DVR.