18.705 Commutative Algebra
Lecture 17

Andrew V. Sutherland
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Recap, and where we are going

Lecture 10: a Noetherian local ring $(R, \mathfrak{m})$ has $\bigcap_n \mathfrak{m}^n = 0$, so R embeds in its completion $\widehat{R} = \varprojlim_n R/\mathfrak{m}^n$, just as $K[x]_{(x)} \subseteq K[[x]]$ and $\mathbb{Z}_{(p)} \subseteq \mathbb{Z}_p$.

Today: I-adic topology and I-adic completion for Noetherian rings and modules, and the tool that makes them work: the Artin–Rees lemma.

- NAK (Lecture 5): if M is a finitely generated R-module and $IM = M$, then there is an $a \in I$ with $am = m$ for all $m \in M$.
- Hilbert basis theorem (Lecture 1): if A is Noetherian then so are $A[x_1, \ldots, x_n]$ and $A[[x_1, \ldots, x_n]]$, and so is every finitely generated A-algebra.
- Graded rings (Lecture 2): $S = \bigoplus_{n \geq 0} S_n$ with $S_m S_n \subseteq S_{m+n}$.
  A graded S-module is an S-module $N = \bigoplus_{n \geq 0} N_n$ with $S_m N_n \subseteq N_{m+n}$.
- Flatness (Lectures 7–8): $- \otimes_R N$ is always right exact, and N is flat iff $J \otimes_R N \to N$ is injective for every finitely generated ideal $J \subseteq R$ (Problem Set 3).

The I-adic filtration and the I-adic topology

Definition

Let I be an ideal of a ring R and let M be an R-module.
The I-adic filtration of M is $M \supseteq IM \supseteq I^2M \supseteq I^3M \supseteq \cdots$.
The I-adic topology on M is the topology with basis $\{m + I^n M : m \in M,\ n \geq 0\}$.

- Elements of $I^n M$ with n large are "close to 0". Let $v(x) := \sup\{n : x \in I^n M\}$ for $x \in M$, and $|x| := e^{-v(x)}$. Then $d(x, y) := |x - y|$ induces the I-adic topology.
  It is a pseudometric: a metric, except that $d(x, y) = 0$ is allowed for $x \neq y$.
  It is nonarchimedean: $|x + y| \leq \max(|x|, |y|)$, so $d(x, z) \leq \max(d(x, y), d(y, z))$.
- M is Hausdorff iff $\bigcap_n I^n M = 0$, iff d is a metric; M is then I-adically separated.
- $R = \mathbb{Z}$, $I = (p)$: a and b are close iff $a \equiv b \bmod p^n$ with n large.
  $R = K[x_1, \ldots, x_n]$, $I = (x_1, \ldots, x_n)$: $v(f)$ is the order of vanishing of f at 0.

The Rees ring and the associated graded ring

Definition

Let I be an ideal of R and t an indeterminate. The Rees ring of I is the graded ring

$$R[It] := \bigoplus_{n\geq 0} I^n t^n = R \oplus It \oplus I^2t^2 \oplus \cdots \subseteq R[t].$$

The associated graded ring of I is the graded ring

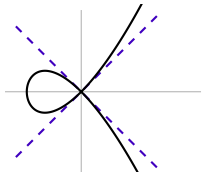
$$\mathrm{gr}_I(R) := \bigoplus_{n\geq 0} I^n/I^{n+1} = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots .$$

- The product of $\bar{a} \in I^m/I^{m+1}$ and $\bar{b} \in I^n/I^{n+1}$ is $\overline{ab} \in I^{m+n}/I^{m+n+1}$ (check!).
- For an R-module M we have the graded $R[It]$-module $M[It] := \bigoplus_n I^n M t^n$, and the graded $\mathrm{gr}_I(R)$-module $\mathrm{gr}_I(M) := \bigoplus_n I^n M/I^{n+1}M$.
- If $I = (f_1, \ldots, f_r)$ then $R[It] = R[f_1 t, \ldots, f_r t]$, and $\mathrm{gr}_I(R) \simeq R[It]/IR[It]$. So if R is Noetherian, then so are $R[It]$ and $\mathrm{gr}_I(R)$.

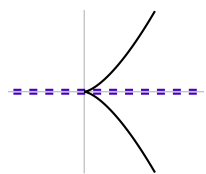
Example: tangent cones

Let $R = K[x_1, \ldots, x_n]$ and $\mathfrak{m} = (x_1, \ldots, x_n)$. Then $\mathfrak{m}^d/\mathfrak{m}^{d+1}$ is the space of forms of degree d, and $\mathrm{gr}_{\mathfrak{m}}(R) \simeq K[x_1, \ldots, x_n]$.

Now let $R = K[x_1, \ldots, x_n]/(f)$ with $f(0) = 0$. Then $\mathrm{gr}_{\mathfrak{m}}(R) \simeq K[x_1, \ldots, x_n]/(f_0)$, where f_0 is the nonzero homogeneous component of f of least degree (check!).
This is the coordinate ring of the tangent cone of $Z(f)$ at the origin.



Node ($\mathrm{char}\, K \neq 2$): $f = y^2 - x^2(x+1)$.
$f_0 = y^2 - x^2 = (y-x)(y+x)$: two lines.



Cusp: $f = y^2 - x^3$.
$f_0 = y^2$: double line.

Stable filtrations

Definition

Let I be an ideal of R. A filtration $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ of an R-module M by submodules is I-consistent if $IM_n \subseteq M_{n+1}$ for all n. It is I-stable if it is I-consistent and for some c we have $IM_n = M_{n+1}$ for all $n \geq c$ (so $M_{c+n} = I^n M_c$).

- The I-adic filtration $\{I^n M\}$ is I-stable. For a submodule $N \subseteq M$, the filtration $\{I^n M \cap N\}$ of N is I-consistent. Is it I-stable?
- If $\{M_n\}$ is I-stable then $I^n M \subseteq M_n$ and $M_{n+c} = I^n M_c \subseteq I^n M$ for all n. So every I-stable filtration of M defines the I-adic topology on M.

Lemma

A filtration $\{M_n\}$ of M is I-consistent iff $\widetilde{M} := \bigoplus_{n\geq 0} M_n t^n$ is an $R[It]$-submodule of $M[t]$. If R is Noetherian and M is finitely generated, then $\{M_n\}$ is I-stable iff $\widetilde{M}$ is a finitely generated $R[It]$-module. (proof on the blackboard)

The Artin–Rees lemma

Theorem (Artin–Rees)

Let R be a Noetherian ring, I an ideal, and $N \subseteq M$ finitely generated R-modules. If $\{M_n\}$ is an I-stable filtration of M, then $\{M_n \cap N\}$ is an I-stable filtration of N.

Proof: To the blackboard!

Corollary

For R, I and $N \subseteq M$ as above, there is a $c \geq 0$ such that for all $n \geq 0$ we have

$$I^{n+c}M \cap N = I^n(I^c M \cap N) \subseteq I^n N.$$

- $I^n N \subseteq I^n M \cap N$ always. Artin–Rees gives the reverse, up to a shift by c. So the I-adic topology on N is the subspace topology induced from M.
- Example: $R = M = K[x]$, $I = (x)$, $N = (x^2)$. For $n \geq 2$ we have $I^n M \cap N = (x^n)$ and $I^n N = (x^{n+2})$, so $c = 2$ works.

On the blackboard: the Artin–Rees lemma

Lemma. Let $\{M_n\}$ be a filtration of M and $\widetilde{M} := \bigoplus_n M_n t^n \subseteq M[t]$. Then (1) $\{M_n\}$ is I-consistent iff $\widetilde{M}$ is an $R[It]$-submodule of $M[t]$. Assume this. Then (2) $\widetilde{M}$ is generated by elements of degree $\leq c$ iff $M_n = I^{n-c}M_c$ for all $n \geq c$; (3) if R is Noetherian and M is finitely generated, $\{M_n\}$ is I-stable iff $\widetilde{M}$ is finitely generated.

Proof. (1) $\widetilde{M}$ is closed under addition and multiplication by R. The ring $R[It]$ is generated by R and It, and $(at)(mt^n) = (am)t^{n+1}$; so $\widetilde{M}$ is an $R[It]$-submodule iff $IM_n \subseteq M_{n+1}$ for all n.
(2) The degree n part of the $R[It]$-submodule generated by $M_0 \oplus M_1 t \oplus \cdots \oplus M_c t^c$ is $\sum_{i \leq c} I^{n-i} M_i t^n$. For $n \geq c$ this is $I^{n-c}M_c t^n$, because $I^{n-i}M_i = I^{n-c}(I^{c-i}M_i) \subseteq I^{n-c}M_c$ for $i \leq c$.
(3) If $\widetilde{M}$ is finitely generated, then the homogeneous components of its generators generate it, in degrees $\leq c$ for some c; apply (2). Conversely, if $M_n = I^{n-c}M_c$ for all $n \geq c$ then $\widetilde{M}$ is generated by $M_0 \oplus \cdots \oplus M_c t^c$, and each $M_i \subseteq M$ is a finitely generated R-module. □

Theorem (Artin–Rees). Let R be Noetherian and $N \subseteq M$ finitely generated R-modules. If $\{M_n\}$ is an I-stable filtration of M, then $\{M_n \cap N\}$ is an I-stable filtration of N.

Proof. Write $I = (f_1, \ldots, f_r)$. Then $R[It] = R[f_1 t, \ldots, f_r t]$ is a finitely generated R-algebra, hence Noetherian, and $\widetilde{M}$ is a finitely generated $R[It]$-module by (3), hence a Noetherian module.
Let $N_n := M_n \cap N$. Then $IN_n \subseteq IM_n \cap N \subseteq N_{n+1}$, so $\widetilde{N} = \bigoplus_n N_n t^n$ is an $R[It]$-submodule of $\widetilde{M}$ by (1). Thus $\widetilde{N}$ is finitely generated, and $\{N_n\}$ is I-stable by (3). For the corollary, take $M_n = I^n M$. □

Presented on the blackboard; not part of the projected slides.

The Krull intersection theorem, again

Theorem (Krull intersection theorem)

Let R be a Noetherian ring, I an ideal, and M a finitely generated R-module. Then there is an $a \in I$ such that $am = m$ for all $m \in \bigcap_n I^n M$.

Proof: Let $N := \bigcap_n I^n M$. By Artin–Rees, $N = I^{c+1}M \cap N \subseteq IN$ for some c. So $IN = N$, and NAK gives a. □

- If R is a domain and $I \neq R$ then $\bigcap_n I^n = 0$, since $1 - a$ is a nonzerodivisor.
- If R is local and $I \neq R$ then $\bigcap_n I^n M = 0$, since $1 - a$ is a unit (cf. Lecture 10).
- The intersection can be nonzero: if $e^2 = e$ and $I = (e)$, then $I^n = I$ for all n.
- Noetherian is needed: let R be the ring of germs of C^∞ functions at $0 \in \mathbb{R}$: C^∞ functions defined near 0, with $f = g$ if they agree on a neighborhood of 0. Then R is local with $\mathfrak{m} = \{f : f(0) = 0\} = (x)$, and $0 \neq e^{-1/x^2} \in \bigcap_n \mathfrak{m}^n$.

Completions

Definition

Let I be an ideal of R and M an R-module. A sequence (m_i) in M is
Cauchy if for every n we have $m_i - m_j \in I^n M$ for all sufficiently large i, j;
a null sequence if for every n we have $m_i \in I^n M$ for all sufficiently large i.
The I-adic completion of M is $\widehat{M}_I := \{\text{Cauchy sequences}\}/\{\text{null sequences}\}$.

- Cauchy sequences in R form a ring, in which the null sequences are an ideal. So $\widehat{R}_I$ is a ring, and similarly $\widehat{M}_I$ is an $\widehat{R}_I$-module (check!).
- Constant sequences give a ring homomorphism $R \to \widehat{R}_I$ and an R-linear map $M \to \widehat{M}_I$ with kernel $\bigcap_n I^n M$ (trivial if M is I-adically separated).
- M is I-adically complete if $M \to \widehat{M}_I$ is an isomorphism.
- For a local ring $(R, \mathfrak{m})$ we call $\widehat{R} := \widehat{R}_{\mathfrak{m}}$ the completion of R, and R is a complete local ring if $R \to \widehat{R}$ is an isomorphism.

Completions as inverse limits

A Cauchy sequence (m_i) is eventually constant modulo $I^n M$, for each n.
Its eventual value $f_n \in M/I^n M$ depends only on its class in $\widehat{M}_I$, and (check!)

$$\widehat{M}_I \;\simeq\; \varprojlim_n M/I^n M := \Big\{(f_n) \in \prod_{n\geq 1} M/I^n M \;:\; f_{n+1} \equiv f_n \bmod I^n M \text{ for all } n\Big\}.$$

Every element of $\widehat{M}_I$ is the sum of a series $g_0 + g_1 + g_2 + \cdots$ with $g_n \in I^n M$.

- $R = A[x_1, \ldots, x_n]$, $I = (x_1, \ldots, x_n)$: R/I^d is the polynomials of degree $< d$, and a compatible sequence of these is a power series. So $\widehat{R}_I \simeq A[[x_1, \ldots, x_n]]$.
- $R = \mathbb{Z}$, $I = (p)$: $\widehat{R}_I = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} =: \mathbb{Z}_p$, the ring of p-adic integers.
  Every element is uniquely $\sum_{i\geq 0} a_i p^i$ with $0 \leq a_i < p$; e.g. $-1 = \sum_{i\geq 0}(p-1)p^i$.
- If $\mathfrak{m}$ is a maximal ideal then $R/\mathfrak{m}^n \simeq R_{\mathfrak{m}}/\mathfrak{m}^n R_{\mathfrak{m}}$, so $\widehat{R}_{\mathfrak{m}} \simeq \widehat{R_{\mathfrak{m}}}$.
  The completion of $K[x]_{(x)}$ is $K[[x]]$, and the completion of $\mathbb{Z}_{(p)}$ is $\mathbb{Z}_p$.

Example: the p-adic integers

An element of $\mathbb{Z}_p$ is a compatible sequence of residues modulo $p, p^2, p^3, \ldots$; equivalently, a series $\sum_{i \geq 0} a_i p^i$ with digits $0 \leq a_i < p$. For example, in $\mathbb{Z}_7$ we have

modulo $7,\ 7^2,\ 7^3,\ 7^4,\ 7^5, \ldots$	digits $a_0, a_1, a_2, a_3, a_4, \ldots$
$2 = (2,\ 2,\ 2,\ 2,\ 2,\ \ldots)$	$2, 0, 0, 0, 0, \ldots$
$2002 = (0,\ 42,\ 287,\ 2002,\ 2002,\ \ldots)$	$0, 6, 5, 5, 0, \ldots$
$-2 = (5,\ 47,\ 341,\ 2399,\ 16805,\ \ldots)$	$5, 6, 6, 6, 6, \ldots$
$2^{-1} = (4,\ 25,\ 172,\ 1201,\ 8404,\ \ldots)$	$4, 3, 3, 3, 3, \ldots$
$\sqrt{2} = (3,\ 10,\ 108,\ 2166,\ 4567,\ \ldots)$	$3, 1, 2, 6, 1, \ldots$
or $(4,\ 39,\ 235,\ 235,\ 12240,\ \ldots)$	$4, 5, 4, 0, 5, \ldots$
$\sqrt[5]{2} = (4,\ 46,\ 95,\ 1124,\ 15530,\ \ldots)$	$4, 6, 1, 3, 6, \ldots$

- $\mathbb{Z} \subseteq \mathbb{Z}_7$ via constant sequences, but $\mathbb{Z}_7$ is much bigger: it is uncountable.
- 2 is a unit in $\mathbb{Z}_7$, but $2002 = 2 \cdot 7 \cdot 11 \cdot 13$ is not: its image in $\mathbb{Z}/7\mathbb{Z}$ is zero.
- 2 has two square roots and one fifth root in $\mathbb{Z}_7$, but no cube root ($x^3 \not\equiv 2 \bmod 7$).

Summary

- The powers of an ideal I define the I-adic topology on every R-module, and two graded rings: the Rees ring $R[It] = \bigoplus_n I^n t^n$ and $\operatorname{gr}_I(R) = \bigoplus_n I^n/I^{n+1}$.
- Artin–Rees: for finitely generated modules $N \subseteq M$ over a Noetherian ring, there is a c with $I^{n+c}M \cap N = I^n(I^c M \cap N)$ for all $n \geq 0$.
- Krull intersection: $\bigcap_n I^n M$ is killed by $1 - a$ for some $a \in I$. It is zero if R is local and $I \neq R$, and $\bigcap_n I^n = 0$ if R is a domain and $I \neq R$.
- The completion $\widehat{M}_I := \{\text{Cauchy sequences}\}/\{\text{null sequences}\} \simeq \varprojlim_n M/I^n M$. $M \to \widehat{M}_I$ has kernel $\bigcap_n I^n M$; M is I-adically complete if it is an isomorphism.
- Examples: the completion of $A[x_1, \ldots, x_n]$ at $(x_1, \ldots, x_n)$ is $A[[x_1, \ldots, x_n]]$, and the completion of $\mathbb{Z}$ at (p) is the ring of p-adic integers $\mathbb{Z}_p$.