18.705 Commutative Algebra
Lecture 16

Andrew V. Sutherland
Adapted from the 2018 lecture notes of Jack Jeffries (§7.3)

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Recap, and where we are going

A domain is normal if it is integrally closed in its fraction field (Lecture 5).
Normality is a local property (Lecture 8), and DVRs are normal (Lecture 15).

Today: a Noetherian domain R is normal iff $R_{\mathfrak{p}}$ is a DVR for every height one prime $\mathfrak{p}$ *and* nonzero principal ideals of R have no embedded primes.

- Lectures 9–10: $\mathrm{Ass}(R/I)$ is the set of primes $(I : r) = \{s \in R : sr \in I,\ r \in R\}$. For R Noetherian it contains $\mathrm{Min}(I)$; the primes in $\mathrm{Ass}(R/I) \setminus \mathrm{Min}(I)$ are the embedded primes of I. There are no embedded primes iff every zerodivisor on R/I lies in a minimal prime of I.
- Krull (Lecture 14): every *minimal* prime of (f) has height at most 1.
- Lecture 15: a DVR is a Noetherian local domain $(V, \mathfrak{m})$ with $\mathfrak{m} = (\pi) \neq 0$.
- Lecture 5: $K[x^2, x^3]$ and $K[x, y, z]/(x^2y - z^2)$ are not normal; $K[x^2, xy, y^2]$ is normal but not a UFD. Today we will see what these examples look like.

The total ring of fractions of a reduced ring

The total ring of fractions $K(R) := W^{-1}R$ inverts the set W of nonzerodivisors of R. We have $R \subseteq K(R)$, and $K(R) = \mathrm{Frac}(R)$ if R is a domain.

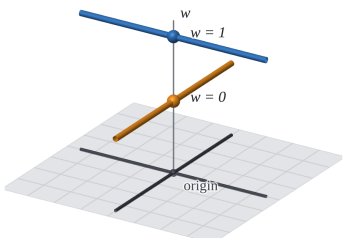
Proposition (proof on the blackboard)

Let R be a reduced Noetherian ring with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.
The set of nonzerodivisors of R is $R \setminus (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n)$, and $K(R) \simeq \prod_i \mathrm{Frac}(R/\mathfrak{p}_i)$.

Example: $R = K[x,y]/(xy)$, two crossing lines.
$K(R) \simeq K(x) \times K(y)$ via $x \mapsto (x,0)$, $y \mapsto (0,y)$.

$e := \frac{x}{x+y} \mapsto (1,0)$ is integral ($e^2 = e$), but $e \notin R$:
a function on the two lines has one value at the origin.

Adjoining e pulls the lines apart: $R[e] \simeq K[x] \times K[y]$.



Normal rings

Definition

A ring R is normal if it is reduced and integrally closed in $K(R)$.

For domains this is the definition of Lecture 5. Note that $K[x]/(x^2)$ is its own total ring of fractions, but it is not reduced.

Proposition (proof on the blackboard)

A Noetherian ring R is normal if and only if it is a finite product of normal domains. In this case $R \simeq R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_n$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the minimal primes of R.

Key point: the idempotents of $K(R) \simeq \prod_i \mathrm{Frac}(R/\mathfrak{p}_i)$ are integral over R.

Corollary

A normal Noetherian local ring is a domain.

Geometrically: distinct irreducible components of a normal variety do not meet.

On the blackboard: total rings of fractions and normal rings

Proposition. Let R be a reduced Noetherian ring with minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, and let W be its set of nonzerodivisors. Then $W = R \setminus \bigcup_i \mathfrak{p}_i$ and $K(R) \simeq \prod_i K_i$, where $K_i := \mathrm{Frac}(R/\mathfrak{p}_i)$.

Proof. $0 = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$ is a minimal primary decomposition, so $\mathrm{Ass}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, and the set of zerodivisors of R is $\bigcup_i \mathfrak{p}_i$ (Lecture 9). The primes of $K(R) = W^{-1}R$ are the $W^{-1}\mathfrak{p}$ with $\mathfrak{p} \subseteq \bigcup_i \mathfrak{p}_i$; by prime avoidance such a $\mathfrak{p}$ lies in some $\mathfrak{p}_i$, so $\mathfrak{p} = \mathfrak{p}_i$. Thus $K(R)$ has exactly n primes $W^{-1}\mathfrak{p}_i$, all maximal, with intersection $W^{-1}(\bigcap_i \mathfrak{p}_i) = 0$. By the CRT, $K(R) \simeq \prod_i K(R)/W^{-1}\mathfrak{p}_i$, and $K(R)/W^{-1}\mathfrak{p}_i$ is a field that is a localization of $R/\mathfrak{p}_i$, so it is K_i. □

Proposition. A Noetherian ring R is normal iff it is a finite product of normal domains.

Proof. ($\Leftarrow$) Let $R = R_1 \times \cdots \times R_n$ with each R_i a normal domain. Then R is reduced, and $(r_1, \dots, r_n)$ is a nonzerodivisor iff every $r_i \neq 0$, so $K(R) = \prod_i \mathrm{Frac}(R_i)$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in K(R)$ is a root of a monic $g \in R[t]$, then α_i is a root of the (monic) image of g in $R_i[t]$, so $\alpha_i \in R_i$ for all i, and $\alpha \in R$.

($\Rightarrow$) R is reduced, so $R \subseteq K(R) = \prod_i K_i$ as above. Let $e_i \in K(R)$ have ith coordinate 1 and all others 0. Then $e_i^2 = e_i$, so e_i is integral over R, hence $e_i \in R$. As $e_i e_j = 0$ for $i \neq j$ and $\sum_i e_i = 1$, we get $R = Re_1 \times \cdots \times Re_n$, and Re_i is the image of R in K_i, which is $R/\mathfrak{p}_i$. If $\alpha \in K_i$ is integral over $R/\mathfrak{p}_i$, say $\alpha^m + \bar{a}_1\alpha^{m-1} + \cdots + \bar{a}_m = 0$ with $a_j \in R$, then $\beta := (0, \dots, \alpha, \dots, 0) \in K(R)$ satisfies $\beta^m + a_1 e_i \beta^{m-1} + \cdots + a_m e_i = 0$, so $\beta \in R$ and $\alpha \in R/\mathfrak{p}_i$. Thus each $R/\mathfrak{p}_i$ is a normal domain. □

Corollary. A normal Noetherian local ring is a domain: a product of $n \geq 2$ nonzero rings is not local.

Presented on the blackboard; not part of the projected slides.

Where can a fraction fail to lie in R?

Let R be a Noetherian domain with fraction field K (so $R \subseteq R_{\mathfrak{p}} \subseteq K$ for all primes $\mathfrak{p}$).
Let $\mathcal{P}$ be the set of primes of R associated to a nonzero principal ideal.

Lemma

$R = \bigcap_{\mathfrak{p} \in \mathcal{P}} R_{\mathfrak{p}}$.
Proof: Let $\alpha = r/u \in K \setminus R$. The ideal $D := \{d \in R : d\alpha \in R\} = (uR : r)$ is proper.
Take $\mathfrak{p} \in \mathrm{Ass}(R/D) \subseteq \mathrm{Ass}(R/uR)$ (multiplication by r embeds R/D in R/uR).
And $\alpha \notin R_{\mathfrak{p}}$: if $\alpha = a/s$ with $s \notin \mathfrak{p}$, then $s \in D \subseteq \mathfrak{p}$. □

Lemma (proof on the blackboard)

For a nonzero prime $\mathfrak{p}$ of R the following are equivalent:

(1) $\mathfrak{p} \in \mathcal{P}$, *that is,* $\mathfrak{p} \in \mathrm{Ass}(R/fR)$ *for some nonzero* $f \in R$;

(2) $\mathfrak{p} \in \mathrm{Ass}(R/gR)$ *for every nonzero* $g \in \mathfrak{p}$;

(3) *there exists* $\alpha \in K \setminus R_{\mathfrak{p}}$ *with* $\mathfrak{p}\alpha \subseteq R_{\mathfrak{p}}$.

Embedded primes of principal ideals

Every height one prime $\mathfrak{p}$ lies in $\mathcal{P}$: it is a minimal prime of (g) for any nonzero $g \in \mathfrak{p}$.
But $\mathcal{P}$ may contain primes of greater height: embedded primes of principal ideals.

Example: $R := K[x^2, x^3, y, xy] \subseteq K[x, y]$ is spanned by the monomials other than x.

- R is a domain of dimension 2, and $\mathfrak{m} := (x^2, x^3, y, xy)$ is a maximal ideal.
- $x \notin R_{\mathfrak{m}}$ but $\mathfrak{m}x \subseteq R$, so $\mathfrak{m} \in \mathcal{P}$: $\mathfrak{m} = (yR : xy)$ is an embedded prime of (y).
- R is not normal: x is a root of $t^2 - x^2 \in R[t]$. Its normalization is $K[x, y]$.
- $R[\frac{1}{y}] = K[x, y, \frac{1}{y}]$ and $R[\frac{1}{x^2}] = K[x, y, \frac{1}{x}]$ are regular, and every prime $\mathfrak{p} \neq \mathfrak{m}$ omits y or x^2, so $R_{\mathfrak{p}}$ is regular for $\mathfrak{p} \neq \mathfrak{m}$, and a DVR when $\operatorname{height} \mathfrak{p} = 1$.

On the blackboard: associated primes of principal ideals

Lemma. Let R be a Noetherian domain with fraction field K and let $\mathfrak{p}$ be a nonzero prime of R. The following are equivalent:
(1) $\mathfrak{p} \in \operatorname{Ass}(R/fR)$ for some nonzero $f \in R$; (2) $\mathfrak{p} \in \operatorname{Ass}(R/gR)$ for every nonzero $g \in \mathfrak{p}$; (3) $\mathfrak{p}\alpha \subseteq R_{\mathfrak{p}}$ for some $\alpha \in K \setminus R_{\mathfrak{p}}$.

Proof. (2) $\Rightarrow$ (1) is clear, since $\mathfrak{p} \neq 0$.

(1) $\Rightarrow$ (3): We have $\mathfrak{p} = (fR : a) = \{r \in R : ra \in fR\}$ for some $a \in R$. Put $\alpha := a/f \in K$. Then $\mathfrak{p}\alpha \subseteq R$ by construction. If $\alpha \in R_{\mathfrak{p}}$, say $a/f = b/s$ with $b \in R$ and $s \in R \setminus \mathfrak{p}$, then $sa = bf \in fR$, so $s \in (fR : a) = \mathfrak{p}$, a contradiction. Thus $\alpha \notin R_{\mathfrak{p}}$.

(3) $\Rightarrow$ (2): Let $S := R_{\mathfrak{p}}$, with maximal ideal $n := \mathfrak{p}S$, so that $n\alpha \subseteq S$ and $\alpha \notin S$. Let $g \in \mathfrak{p}$ be nonzero, and put $b := g\alpha$. Then $b \in S$, since $g \in n$ and $n\alpha \subseteq S$; $b \notin gS$, since otherwise $\alpha = b/g \in S$; and $nb = g(n\alpha) \subseteq gS$. So the image of b in S/gS is a nonzero element annihilated by n; its annihilator is a proper ideal containing the maximal ideal n, so it equals n, and $n \in \operatorname{Ass}_S(S/gS)$. Associated primes localize (Lecture 9): $\operatorname{Ass}_{R_{\mathfrak{p}}}\left((R/gR)_{\mathfrak{p}}\right) = \{qR_{\mathfrak{p}} : q \in \operatorname{Ass}_R(R/gR),\ q \subseteq \mathfrak{p}\}$, so $\mathfrak{p} \in \operatorname{Ass}_R(R/gR)$. □

Example. $R = K[x^2, x^3, y, xy]$, $m = (x^2, x^3, y, xy)$, $\alpha = x$: if $x = a/s$ with $a, s \in R$ and $s \notin m$, then $\frac{\partial}{\partial x}(sx)(0,0) = s(0,0) \neq 0$, but $sx = a \in R$. So $x \notin R_m$, while $mx \subseteq R$. Taking $g = y$ gives $b = xy$: $xy \notin yR$ and $mxy \subseteq yR$.

Presented on the blackboard; not part of the projected slides.

Serre's criterion for normality

Theorem (Serre)

Let R be a Noetherian domain. The following are equivalent:

(1) *R is normal;*

(2) *$R_{\mathfrak{p}}$ is a DVR for every prime $\mathfrak{p}$ associated to a nonzero principal ideal;*

(3) (R_1) *$R_{\mathfrak{p}}$ is a DVR for every prime $\mathfrak{p}$ of height one, and*
(S_2) *every prime associated to a nonzero principal ideal has height one.*

Proof: To the blackboard!

- (R_1): regular in codimension one. (S_2): principal ideals have no embedded primes.
- $K[x^2, x^3, y, xy]$ satisfies only (R_1); the Whitney umbrella satisfies only (S_2).
- For the generalization to reduced Noetherian rings, see Jeffries (Theorem 7.27).

On the blackboard: Serre's criterion

Theorem. For a Noetherian domain R with fraction field K, the following are equivalent: (1) R is normal; (2) $R_\mathfrak{p}$ is a DVR for all $\mathfrak{p} \in \mathcal{P}$; (3) $R_\mathfrak{p}$ is a DVR for every height one prime $\mathfrak{p}$, and every $\mathfrak{p} \in \mathcal{P}$ has height one.

Proof. (1) $\Rightarrow$ (2): Let $\mathfrak{p} \in \mathcal{P}$ and $S := R_\mathfrak{p}$, a normal Noetherian local domain (Lecture 8) with maximal ideal $\mathfrak{n} := \mathfrak{p}S \neq 0$. By the lemma there is an $\alpha \in K \setminus S$ with $\mathfrak{n}\alpha \subseteq S$. Now $\mathfrak{n}\alpha$ is an ideal of S, so either $\mathfrak{n}\alpha \subseteq \mathfrak{n}$ or $\mathfrak{n}\alpha = S$.

Suppose $\mathfrak{n}\alpha \subseteq \mathfrak{n}$, and let $\mathfrak{n} = (x_1, \ldots, x_m)$. Then $\alpha x_i = \sum_j a_{ij} x_j$ with $a_{ij} \in S$, so the matrix $B := \alpha I - (a_{ij})$ kills the vector $(x_1, \ldots, x_m)$, and multiplying by $\operatorname{adj}(B)$ shows that $\det(B)x_i = 0$ for all i (the determinant trick). As $\mathfrak{n} \neq 0$ and K is a field, $\det(B) = 0$, which is a monic equation for α with coefficients in S. But S is normal, so $\alpha \in S$, a contradiction.

Therefore $\mathfrak{n}\alpha = S$, so $\pi\alpha = 1$ for some $\pi \in \mathfrak{n}$. For every $x \in \mathfrak{n}$ we then have $x = \pi(\alpha x) \in \pi S$, since $\alpha x \in S$. Thus $\mathfrak{n} = (\pi)$ is principal and nonzero, and S is a DVR (Lecture 15). The uniformizer is the inverse of the "bad fraction" α.

(2) $\Rightarrow$ (1): Let $\alpha \in K$ be integral over R. For each $\mathfrak{p} \in \mathcal{P}$, α is integral over the DVR $R_\mathfrak{p}$, which is normal (Lecture 15), so $\alpha \in R_\mathfrak{p}$. Hence $\alpha \in \bigcap_{\mathfrak{p} \in \mathcal{P}} R_\mathfrak{p} = R$ by the first lemma.

(2) $\Rightarrow$ (3): Every height one prime lies in $\mathcal{P}$; if $\mathfrak{p} \in \mathcal{P}$ then $\operatorname{height} \mathfrak{p} = \dim R_\mathfrak{p} = 1$, since $R_\mathfrak{p}$ is a DVR.

(3) $\Rightarrow$ (2): Immediate. $\square$

Presented on the blackboard; not part of the projected slides.

Consequences of Serre's criterion

Corollary

Let R be a Noetherian normal domain.

(1) *If $f \in R$ is a nonzero nonunit, then every associated prime of (f) has height one.*

(2) $R = \bigcap_{\text{height } \mathfrak{p}=1} R_{\mathfrak{p}}$.

(3) *If $\mathfrak{p}$ is a prime of R and $R_{\mathfrak{p}}$ is not regular, then* $\text{height } \mathfrak{p} \geq 2$.

(1) strengthens Krull's theorem: (f) has no embedded primes.
(2) is algebraic Hartogs: a rational function with no poles in codimension one is regular.
(3) says that the singular locus of a normal variety has codimension at least 2.

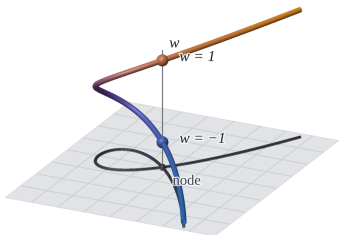
Proposition

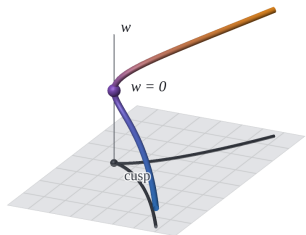
A Noetherian domain R with $R_{\mathfrak{m}}$ regular for every maximal ideal $\mathfrak{m}$ is normal.

Proof: Regular local rings are UFDs (Lecture 15), hence normal (Lecture 5), and normality is a local property (Lecture 8). □

Normal curves

Corollary: For Noetherian local domains of dimension 1: normal $\Leftrightarrow$ DVR $\Leftrightarrow$ regular.
So for $K = \overline{K}$, an irreducible affine curve is nonsingular iff its coordinate ring is normal.





Node ($\operatorname{char} K \neq 2$):
$R = K[x,y]/(y^2 - x^2(x+1))$.
$w := y/x \notin R$ satisfies $w^2 = x + 1$.
$R[w] = K[w]$: two points over the node.

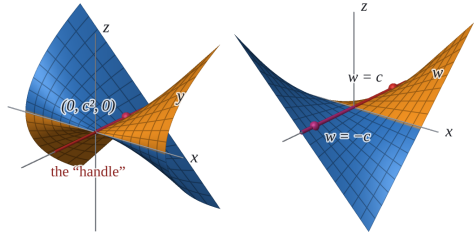
Cusp:
$R = K[x,y]/(y^2 - x^3) \simeq K[w^2, w^3]$.
$w := y/x \notin R$ satisfies $w^2 = x$.
$R[w] = K[w]$: one point over the cusp.

The singular curve is the shadow of a smooth curve in 3-space. [interactive version]

Normal surfaces

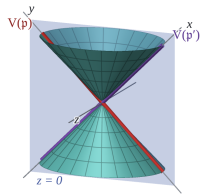
Whitney umbrella: $R = K[x, y, z]/(x^2y - z^2)$.

- Singular along the line $x = z = 0$: $R_{\mathfrak{p}}$ is not regular for $\mathfrak{p} = (x, z)$.
- So (R_1) fails; (S_2) holds (not proved).
- $w := z/x$ is integral: $w^2 = y$.
- $R[w] = K[x, w]$: a plane, folded onto the umbrella by $(x, w) \mapsto (x, w^2, xw)$.



Quadric cone: $R = K[x, y, z]/(z^2 - xy) \simeq K[s^2, t^2, st]$.

- R is normal (Lecture 5), but not regular.
- The only singular point is the vertex.
- A normal surface has only isolated singular points.



Orders of vanishing

Let R be a Noetherian normal domain with fraction field K. Each height one prime $\mathfrak{p}$ gives a DVR $R_\mathfrak{p}$ with valuation $\operatorname{ord}_\mathfrak{p}\colon K^\times \to \mathbb{Z}$, the order of vanishing along $V(\mathfrak{p})$.

Theorem

If $f \in R$ is a nonzero nonunit with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ and $e_i := \operatorname{ord}_{\mathfrak{p}_i}(f)$, then

$$(f) = \mathfrak{p}_1^{(e_1)} \cap \cdots \cap \mathfrak{p}_t^{(e_t)}, \qquad \text{where } \mathfrak{p}^{(e)} := \mathfrak{p}^e R_\mathfrak{p} \cap R = \{r \in R : \operatorname{ord}_\mathfrak{p}(r) \geq e\} \cup \{0\}.$$

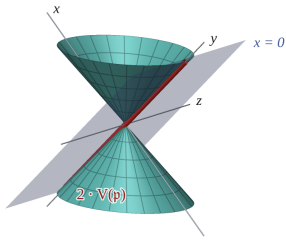
Proof: (f) has no embedded primes, so its primary components are the $fR_{\mathfrak{p}_i} \cap R$ (second uniqueness theorem, Lecture 10), and $fR_{\mathfrak{p}_i} = \mathfrak{p}_i^{e_i} R_{\mathfrak{p}_i}$ in the DVR $R_{\mathfrak{p}_i}$. □

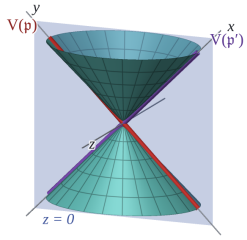
- $\alpha \in K^\times$ lies in R iff $\operatorname{ord}_\mathfrak{p}(\alpha) \geq 0$ for all height one $\mathfrak{p}$ (algebraic Hartogs).
- For nonzero $f, g \in R$: f divides g iff $\operatorname{ord}_\mathfrak{p}(f) \leq \operatorname{ord}_\mathfrak{p}(g)$ for all height one $\mathfrak{p}$.
- So f is determined up to a unit by its orders of vanishing.

Example: lines on the cone

Let $R = K[x,y,z]/(z^2 - xy)$, $\mathfrak{p} := (x,z)$, $\mathfrak{p}' := (y,z)$.
Height one primes: $R/\mathfrak{p} = K[y]$, $R/\mathfrak{p}' = K[x]$.

- In $R_{\mathfrak{p}}$, y is a unit and $x = z^2/y$, so $\mathfrak{p}R_{\mathfrak{p}} = (z)$: $\mathrm{ord}_{\mathfrak{p}}(z) = 1$, $\mathrm{ord}_{\mathfrak{p}}(x) = 2$, $\mathrm{ord}_{\mathfrak{p}}(y) = 0$.
- $(z) = \mathfrak{p} \cap \mathfrak{p}'$: $z = 0$ cuts the cone in two lines.
- $(x) = \mathfrak{p}^{(2)}$: $x = 0$ is tangent along $V(\mathfrak{p})$.
- $\mathfrak{p}^2 = x \cdot (x,y,z) \subsetneq \mathfrak{p}^{(2)}$: the ordinary power has an embedded component at the vertex.
- $\mathfrak{p}$ is not principal, so R is not a UFD.





Divisors and Dedekind domains

Let R be a Noetherian normal domain. The homomorphism $\operatorname{div}\colon K^\times \to \bigoplus_{\text{height } \mathfrak{p}=1} \mathbb{Z}$ defined by $\alpha \mapsto (\operatorname{ord}_\mathfrak{p}(\alpha))_\mathfrak{p}$ has kernel $R^\times$; its cokernel is the divisor class group $\operatorname{Cl}(R)$.

- $\operatorname{Cl}(R) = 0$ iff every height one prime is principal iff R is a UFD (Problem Set 5).
- For the cone: $\operatorname{div}(x) = 2[\mathfrak{p}]$, $\operatorname{div}(z) = [\mathfrak{p}] + [\mathfrak{p}']$, and $\operatorname{Cl}(R) \simeq \mathbb{Z}/2\mathbb{Z}$.

A Dedekind domain is a Noetherian normal domain of dimension at most one.
Examples: PIDs, $\mathcal{O}_K$ for a number field K, coordinate rings of nonsingular curves.

Theorem

Every nonzero proper ideal I of a Dedekind domain R has a unique factorization $I = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}$ into powers of distinct primes.

Proof: The associated primes $\mathfrak{p}_i$ of I are maximal, so the primary components of I are $IR_{\mathfrak{p}_i} \cap R = \mathfrak{p}_i^{e_i} R_{\mathfrak{p}_i} \cap R = \mathfrak{p}_i^{e_i}$ (as $\mathfrak{p}_i$ is maximal), and $\bigcap_i \mathfrak{p}_i^{e_i} = \prod_i \mathfrak{p}_i^{e_i}$ by the CRT. Uniqueness: any such product is a minimal primary decomposition of I. $\square$

Summary

- A reduced Noetherian ring R has total ring of fractions $\prod_{\mathfrak{p} \in \mathrm{Min}(R)} \mathrm{Frac}(R/\mathfrak{p})$.
  A Noetherian ring is normal iff it is a finite product of normal domains.

- Serre's criterion: a Noetherian domain R is normal iff (R_1) $R_\mathfrak{p}$ is a DVR for every height one prime $\mathfrak{p}$, and (S_2) nonzero principal ideals have no embedded primes.

- Regular $\Rightarrow$ normal $\Rightarrow$ singular locus of codimension at least 2.
  In dimension one, normal $\Leftrightarrow$ regular (think of the node and the cusp).

- In a Noetherian normal domain, $R = \bigcap_{\mathrm{height}\,\mathfrak{p}=1} R_\mathfrak{p}$ and $(f) = \bigcap_i \mathfrak{p}_i^{(e_i)}$:
  f is determined up to a unit by its orders of vanishing $e_i = \mathrm{ord}_{\mathfrak{p}_i}(f)$.

- Ideals in Dedekind domains (Noetherian normal, dimension ≤ 1) factor uniquely.
  The class group measures the failure of unique factorization of elements.