18.705 Commutative Algebra
Lecture 15

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Recap, and where we are going

A K-algebra generated by n elements has dimension at most n, with equality iff it is a polynomial ring (Lecture 12). The local analogue (Lecture 14): a Noetherian local ring $(R, \mathfrak{m}, k)$ has $\dim R \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$. Today we study local rings where equality holds.

- NAK (Lecture 5): $\mathfrak{m}$ is minimally generated by $\dim_k \mathfrak{m}/\mathfrak{m}^2$ elements, and $x_1, \ldots, x_m \in \mathfrak{m}$ are part of a minimal generating set iff their images in $\mathfrak{m}/\mathfrak{m}^2$ are k-linearly independent.
- Krull (Lecture 14): minimal primes of $(f_1, \ldots, f_n)$ have height $\leq n$; $x_1, \ldots, x_m \in \mathfrak{m}$ are parameters iff $\dim R/(x_1, \ldots, x_m) = \dim R - m$, and $\dim R/(x_1, \ldots, x_m) \geq \dim R - m$ always.
- If R is an affine domain of dimension d then $\dim R_{\mathfrak{m}} = d$ for every maximal ideal $\mathfrak{m}$, and $\operatorname{height}(\mathfrak{p}) = d - \dim R/\mathfrak{p}$ for every prime $\mathfrak{p}$ (Lecture 12).
- Lecture 5: for the ideal I of 2×2 minors of $\left(\begin{smallmatrix} u & v & w \\ x & y & z \end{smallmatrix}\right)$ in $K[u, v, w, x, y, z]$, $\mu(IR_{\mathfrak{m}}) = 3$ at $\mathfrak{m} := (u, v, w, x, y, z)$, but $\mu(IR_{\mathfrak{n}}) = 2$ at $\mathfrak{n} := (u - 1, v, w, x, y, z)$ and $\mu(IR_{\mathfrak{p}}) = 2$ at $\mathfrak{p} := (x, y, z)$.

Regular local rings

Definition

A Noetherian local ring $(R, \mathfrak{m}, k)$ is regular if $\dim R = \dim_k \mathfrak{m}/\mathfrak{m}^2$, so $\mathfrak{m}$ is generated by $\dim R$ elements. Such a generating set is called a regular system of parameters.

- $\dim R = 0$: R is regular iff $\mathfrak{m} = 0$ iff R is a field.
- $\dim R = 1$: R is regular iff $\mathfrak{m}$ is principal. So $K[x]_{(x)}$, $K[[x]]$, $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$ are regular, but $K[[x^2, x^3]]$ is not ($\mathfrak{m} = (x^2, x^3)$), nor are the local rings $K[x, y]_{(x,y)}/(y^2 - x^3)$ and $K[x, y]_{(x,y)}/(y^2 - x^2(x + 1))$ of the cuspidal and nodal cubics at the origin: $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2 > 1$.
- For $a \in K^n$, $K[x_1, \ldots, x_n]_{\mathfrak{m}_a}$ is regular: $\dim = n$ and $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$. In fact $K[x_1, \ldots, x_n]_{\mathfrak{p}}$ is regular for every prime $\mathfrak{p}$ (lemma below). So is $\mathbb{Z}[x]_{(p,x)}$, of dimension 2.
- Regular local rings turn out to be domains (even UFDs, a hard theorem); $K[x, y, z]_{(x,y,z)}/(z^2 - xy)$ is a domain of dimension 2 that is not regular.

Example: the rank-one locus

Let $R := K[u,v,w,x,y,z]/I$ with $I := (uy - vx,\ uz - wx,\ vz - wy)$ the ideal of 2×2 minors of $\left(\begin{smallmatrix} u & v & w \\ x & y & z \end{smallmatrix}\right)$, a prime of height 2, so R is a domain of dimension 4 (recall that $\operatorname{height}(I) + \dim A/I = \dim A$ for every proper ideal I of an affine domain A, Lecture 12).

- At the origin $\mathfrak{m} := (u,v,w,x,y,z)$: $I \subseteq \mathfrak{m}^2$, so $\mathfrak{m}R_{\mathfrak{m}}/\mathfrak{m}^2R_{\mathfrak{m}}$ has basis u,v,w,x,y,z and dimension $6 > 4 = \dim R_{\mathfrak{m}}$. $R_{\mathfrak{m}}$ *is not regular.*
- At $\mathfrak{n} := (u-1,v,w,x,y,z)$: u is a unit in $R_{\mathfrak{n}}$, and $uy = vx$, $uz = wx$ give $y = \frac{v}{u}x$ and $z = \frac{w}{u}x$, so $\mathfrak{n}R_{\mathfrak{n}} = (u-1,v,w,x)$ is generated by $4 = \dim R_{\mathfrak{n}}$ elements. $R_{\mathfrak{n}}$ *is regular.*
- At the prime $\mathfrak{p} := (v,w,x,y,z)$ (the line of matrices $\left(\begin{smallmatrix} u & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$): $R/\mathfrak{p} \simeq K[u]$, so $\dim R_{\mathfrak{p}} = \operatorname{height}(\mathfrak{p}) = 4 - 1 = 3$, and as above $\mathfrak{p}R_{\mathfrak{p}} = (v,w,x)$. $R_{\mathfrak{p}}$ *is regular.*
- Compare Lecture 5: $\mu(IR_{\mathfrak{m}}) = 3$ but $\mu(IR_{\mathfrak{n}}) = \mu(IR_{\mathfrak{p}}) = 2 = \operatorname{height}(I)$. The regular points are those where $Z_K(I)$ is locally cut out by $\operatorname{height}(I)$ equations; these are the points where the Jacobian matrix of I has rank $\operatorname{height}(I)$.

Regular local rings are domains

Theorem

A regular local ring is a domain.
Proof: To the blackboard!

Definition

Elements $x_1, \ldots, x_m \in \mathfrak{m}$ of a regular local ring $(R, \mathfrak{m}, k)$ are regular parameters if their images in $\mathfrak{m}/\mathfrak{m}^2$ are k-linearly independent (so part of a regular system of parameters).

Corollary

If $x_1, \ldots, x_m$ are regular parameters of R, then $R/(x_1, \ldots, x_m)$ is a regular local ring of dimension $\dim R - m$, and $(x_1, \ldots, x_m)$ is a prime ideal of height m. In particular, regular parameters are parameters.

Parameters need not be regular parameters: x^2 is a parameter of $K[x]_{(x)}$, but $x^2 \in \mathfrak{m}^2$, and $K[x]_{(x)}/(x^2)$ is not regular (nor a domain).

On the blackboard: regular local rings are domains

Theorem. A regular local ring $(R, \mathfrak{m}, k)$ is a domain.

Proof. Induct on $d := \dim R$. If $d = 0$ then $\mathfrak{m} = 0$ and $R = k$. Let $d \geq 1$. Then $\mathfrak{m} \neq \mathfrak{m}^2$ (NAK), and $\mathfrak{m}$ is not a minimal prime of R (as $d > 0$), so prime avoidance gives $x \in \mathfrak{m}$ with $x \notin \mathfrak{m}^2$ and $x \notin \mathfrak{p}$ for each of the finitely many minimal primes $\mathfrak{p}$ of R. Let $\bar{R} := R/xR$, with maximal ideal $\bar{\mathfrak{m}}$. Then $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 = \mathfrak{m}/(\mathfrak{m}^2 + xR)$ has k-dimension $d - 1$, as the image of x in $\mathfrak{m}/\mathfrak{m}^2$ is nonzero. Thus $\dim \bar{R} \leq d - 1$ by Krull, while $\dim \bar{R} \geq d - 1$ (Lecture 14), so $\bar{R}$ is regular of dimension $d - 1$, hence a domain by induction: xR is a prime ideal.

Since x lies in no minimal prime, xR is not minimal, so there is a prime $\mathfrak{p} \subsetneq xR$, and $x \notin \mathfrak{p}$ (otherwise $xR \subseteq \mathfrak{p}$). For $y \in \mathfrak{p}$ write $y = ax$ with $a \in R$; then $a \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $x \notin \mathfrak{p}$. Hence $\mathfrak{p} = x\mathfrak{p} \subseteq \mathfrak{m}\mathfrak{p}$, and NAK gives $\mathfrak{p} = 0$. So 0 is a prime ideal and R is a domain. □

Corollary. If $x_1, \ldots, x_m$ are regular parameters of R then $R/(x_1, \ldots, x_m)$ is regular of dimension $d - m$ and $(x_1, \ldots, x_m)$ is a prime of height m.

Proof. For $m = 1$: $x := x_1 \notin \mathfrak{m}^2$, so as in the proof above R/xR is regular of dimension $d - 1$ and xR is prime; and $\text{height}(xR) = 1$, since $0 \subsetneq xR$ (R is a domain) and $\text{height}(xR) \leq 1$ by Krull. In general, $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2 = \mathfrak{m}/(\mathfrak{m}^2 + x_1 R)$ is the quotient of $\mathfrak{m}/\mathfrak{m}^2$ by the span of $\bar{x}_1$, so the images of $x_2, \ldots, x_m$ in $\bar{R} := R/x_1R$ are regular parameters of the regular local ring $\bar{R}$, and induction on m shows that $R/(x_1, \ldots, x_m)$ is regular of dimension $d - m$ and that each $(x_1, \ldots, x_i)$ is prime. Since $x_i \notin (x_1, \ldots, x_{i-1})$ (by linear independence in $\mathfrak{m}/\mathfrak{m}^2$), the primes $0 \subsetneq (x_1) \subsetneq \cdots \subsetneq (x_1, \ldots, x_m)$ form a chain of length m, so $\text{height}(x_1, \ldots, x_m) \geq m$, with equality by Krull. □

Presented on the blackboard; not part of the projected slides.

Tangent spaces

Let $S := K[x_1, \ldots, x_n]$ and $a \in K^n$, with $\mathfrak{n} := \mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$.
For $f \in S$ put $\nabla_a f := \sum_j \frac{\partial f}{\partial x_j}(a)\,(x_j - a_j)$; expanding f in powers of the $x_j - a_j$ gives

$$f = f(a) + \nabla_a f + (\text{an element of } \mathfrak{n}^2).$$

So $\mathfrak{n}/\mathfrak{n}^2$ has K-basis $x_1 - a_1, \ldots, x_n - a_n$. For $f \in \mathfrak{n}$ the image of f in $\mathfrak{n}/\mathfrak{n}^2$ is $\nabla_a f$.

Now let $K = \overline{K}$, let $\mathfrak{p} = (f_1, \ldots, f_m) \subseteq \mathfrak{n}$ be a prime of height h, $X := Z_K(\mathfrak{p})$, and let $R := (S/\mathfrak{p})_{\mathfrak{n}}$ be the local ring of X at a, with maximal ideal $\mathfrak{m} = \mathfrak{n}R$. Then $\dim R = \dim X = n - h$, and $g f_i \equiv g(a) \nabla_a f_i \bmod \mathfrak{n}^2$ shows that the image of $\mathfrak{p}$ in $\mathfrak{n}/\mathfrak{n}^2$ is spanned by the $\nabla_a f_i$. Therefore

$$\dim_K \mathfrak{m}/\mathfrak{m}^2 = \dim_K \mathfrak{n}/(\mathfrak{n}^2 + \mathfrak{p}) = n - \operatorname{rank} J(a) = \dim_K T_a X.$$

Here $J(a) := \left(\frac{\partial f_i}{\partial x_j}(a)\right)$ is the Jacobian matrix at a and $T_a X := \ker J(a) \subseteq K^n$ is the tangent space of X at a; its dual $\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space.

Tangent spaces

As on the previous slide: $K = \overline{K}$, $\mathfrak{p} = (f_1, \ldots, f_m) \subseteq \mathfrak{n}$ is a prime of height h, $X := Z_K(\mathfrak{p})$, $R := (S/\mathfrak{p})_{\mathfrak{n}}$ is the local ring at a with maximal ideal $\mathfrak{m} = \mathfrak{n}R$, $J(a) := (\frac{\partial f_i}{\partial x_j}(a))$, and $T_aX := \ker J(a) \subseteq K^n$.

Proposition

The following are equivalent: R *is regular;* $\dim T_aX = \dim X$; $\operatorname{rank} J(a) = h$; $a \notin Z_K(I_h(J))$, *with* $I_h(J)$ *the ideal of* $h \times h$ *minors of* $J := (\partial f_i/\partial x_j)$.

Proof: By the previous slide, $\dim_K \mathfrak{m}/\mathfrak{m}^2 = \dim_K T_aX = n - \operatorname{rank} J(a)$ and $\dim R = \dim X = n - h$, so the first three conditions are equivalent. Since $\dim_K \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$ (Lecture 14), $\operatorname{rank} J(a) \leq h$, with equality iff some $h \times h$ minor of $J(a)$, i.e. some generator of $I_h(J)$ evaluated at a, is nonzero, i.e. iff $a \notin Z_K(I_h(J))$.

- The points $a \in X$ where R is regular are the nonsingular points of X; the singular locus $X \cap Z_K(I_h(J))$ is closed. At every $a \in X$ we have $\dim T_aX \geq \dim X$.
- T_aX is cut out by linear $\nabla_a f$ for $f \in \mathfrak{p}$, so depends only on $\mathfrak{p}$; pairing $\nabla_a f$ with vectors identifies $\mathfrak{n}/\mathfrak{n}^2$ with $(K^n)^*$ and T_aX with the dual of $\mathfrak{n}/(\mathfrak{n}^2 + \mathfrak{p}) = \mathfrak{m}/\mathfrak{m}^2$.

The Jacobian criterion

Over an arbitrary field K, and at an arbitrary prime $\mathfrak{q} \supseteq \mathfrak{p}$, one direction of the proposition survives:

Lemma

For every prime $\mathfrak{q}$ of $S = K[x_1, \ldots, x_n]$, the local ring $S_{\mathfrak{q}}$ is regular.

Theorem (Jacobian criterion)

Let $\mathfrak{p} = (f_1, \ldots, f_m)$ be a prime of S of height h, let $\mathfrak{q} \supseteq \mathfrak{p}$ be a prime, and let $J := (\partial f_i / \partial x_j)$. If $I_h(J) \not\subseteq \mathfrak{q}$ then $(S/\mathfrak{p})_{\mathfrak{q}}$ is regular.
Proof: To the blackboard!

The converse holds for K perfect (and for $K = \overline{K}$ and $\mathfrak{q}$ maximal, as above), but not in general: for $K = \mathbb{F}_p(t)$ and $\mathfrak{p} = (x^p - t) \subseteq K[x]$, the ring $(S/\mathfrak{p})_{\mathfrak{p}} = K(t^{1/p})$ is a field, hence regular, but $J = (px^{p-1}) = 0$.

On the blackboard: the Jacobian criterion

Lemma. For every prime $\mathfrak{q}$ of $S = K[x_1, \dots, x_n]$, $S_{\mathfrak{q}}$ is regular.

Proof. Induct on n; for $n = 0$, $S = K$. Let $S' := K[x_1, \dots, x_{n-1}]$, $\mathfrak{q}' := \mathfrak{q} \cap S'$, and $A := S'_{\mathfrak{q}'}$, regular by induction, with $\mathfrak{q}'A = (g_1, \dots, g_r)$ and $r = \dim A$. As $S' \setminus \mathfrak{q}' \subseteq S \setminus \mathfrak{q}$, $S_{\mathfrak{q}} = A[x_n]_Q$ for the prime $Q := \mathfrak{q}A[x_n]$, and $Q \cap A = \mathfrak{q}'A$. Now $A[x_n]/\mathfrak{q}'A[x_n] = \kappa[x_n]$ with $\kappa := A/\mathfrak{q}'A$ a field, so $Q/\mathfrak{q}'A[x_n]$ is 0 or generated by one polynomial; lifting it, $Q = (g_1, \dots, g_r)$ or $Q = (g_1, \dots, g_r, g)$, so $\mu(QS_{\mathfrak{q}}) \le r$ or $r + 1$. Conversely, a chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{q}'A$ in A gives a chain $\mathfrak{p}_0 A[x_n] \subsetneq \cdots \subsetneq \mathfrak{q}'A[x_n]$ of primes of $A[x_n]$ (as $A[x_n]/\mathfrak{p}_i A[x_n] = (A/\mathfrak{p}_i)[x_n]$ is a domain) ending at Q in the first case and extended by Q in the second, so $\dim S_{\mathfrak{q}} = \operatorname{height}(Q) \ge r$ or $r + 1$. By Krull, $\dim S_{\mathfrak{q}} = \mu(QS_{\mathfrak{q}})$. □

Theorem. Let $\mathfrak{p} = (f_1, \dots, f_m) \subseteq \mathfrak{q}$ be primes of S with $\operatorname{height}(\mathfrak{p}) = h$. If $I_h(J) \not\subseteq \mathfrak{q}$ then $(S/\mathfrak{p})_{\mathfrak{q}}$ is regular.

Proof. Reorder so that $\Delta := \det(\partial f_i/\partial x_j)_{i,j \le h} \notin \mathfrak{q}$. *Claim:* $f_1, \dots, f_h$ *are regular parameters of* $S_{\mathfrak{q}}$, i.e. their images in $\mathfrak{q}S_{\mathfrak{q}}/\mathfrak{q}^2S_{\mathfrak{q}}$ are linearly independent over $\kappa(\mathfrak{q}) := S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}}$. Suppose $\sum_{i \le h} s_i f_i \in \mathfrak{q}^2 S_{\mathfrak{q}}$ with $s_i \in S_{\mathfrak{q}}$. Clearing denominators, there is an $s \in S \setminus \mathfrak{q}$ with $ss_i \in S$ and $\sum_i (ss_i) f_i = \sum_k q_k q'_k$ in S for some $q_k, q'_k \in \mathfrak{q}$. Applying $\partial_j := \partial/\partial x_j$ gives
$\sum_i \partial_j(ss_i) f_i + \sum_i (ss_i)\, \partial_j f_i = \sum_k (\partial_j q_k) q'_k + q_k (\partial_j q'_k) \in \mathfrak{q}$, and $f_i \in \mathfrak{q}$, so $\sum_i (ss_i)\, \partial_j f_i \in \mathfrak{q}$ for every j. For $j \le h$ this says that $(ss_1, \dots, ss_h)$ reduced modulo $\mathfrak{q}$ is killed by the matrix $(\partial_j f_i)_{i,j \le h}$ over $\kappa(\mathfrak{q})$, which is invertible as $\Delta \notin \mathfrak{q}$. So $ss_i \in \mathfrak{q}$, and as $s \notin \mathfrak{q}$, $s_i \in \mathfrak{q}S_{\mathfrak{q}}$ for all i, proving the claim.

By the lemma $S_{\mathfrak{q}}$ is regular, so by the corollary $P := (f_1, \dots, f_h)S_{\mathfrak{q}}$ is a prime of height h. It is contained in the prime $\mathfrak{p}S_{\mathfrak{q}}$, which also has height h, so $P = \mathfrak{p}S_{\mathfrak{q}}$ and $(S/\mathfrak{p})_{\mathfrak{q}} = S_{\mathfrak{q}}/P$ is regular. □

Presented on the blackboard; not part of the projected slides.

Examples

- Let $\operatorname{char} K \neq 2$. For $R := K[x,y,z]/(x^2+y^2+z^2-1)$ we have $J = (2x \;\; 2y \;\; 2z)$, and no prime of R contains $(x,y,z) \ni x^2+y^2+z^2 = 1$, so $R_{\mathfrak{q}}$ is regular for every prime $\mathfrak{q}$ (cf. the tangent fields on S^2 in Lecture 7).

- For the cone $R := K[x,y,z]/(x^2+y^2+z^2)$, $R_{\mathfrak{q}}$ is regular for $\mathfrak{q} \neq (x,y,z)$, and $R_{(x,y,z)}$ is not: it has dimension 2 but $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 3$.

- For the rank-one locus $R = K[u,v,w,x,y,z]/I$, the 2×2 minors of the 3×6 Jacobian J of the three minors include $u^2, \ldots, z^2$ (check!), so $I_2(J) \not\subseteq \mathfrak{q}$ and $R_{\mathfrak{q}}$ is regular for every prime $\mathfrak{q} \neq (u,v,w,x,y,z)$ (see blackboard slides).

- $A := \mathbb{Z}[x,y]_{(2,x,y)}$ is regular of dimension 3: $0 \subsetneq (2) \subsetneq (2,x) \subsetneq (2,x,y) = \mathfrak{m}$. So $A/(2-x^2+y^2)$ is regular, as $2-x^2+y^2 \notin \mathfrak{m}^2$ is a regular parameter. But $A/(4-x^2+y^2)$ is not: it has dimension 2 while $4-x^2+y^2 \in \mathfrak{m}^2$ leaves $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 3$.

On the blackboard: the ideal of the rank-one locus is prime

Claim. For every field K, the ideal $I = (uy - vx,\ uz - wx,\ vz - wy)$ of $S := K[u, v, w, x, y, z]$ is a prime of height 2.

Proof. Let $\varphi\colon S \to K[u, v, w, t]$ be the K-algebra homomorphism that fixes u, v, w and sends x, y, z to tu, tv, tw. Then $P := \ker \varphi$ is prime, and $I \subseteq P$, since φ kills the three minors. We show that $P \subseteq I$.

Call a monomial of S *standard* if it is not divisible by vx, wx, or wy. Modulo I we have $vx \equiv uy$, $wx \equiv uz$, $wy \equiv vz$, and each of these replacements lowers $b + 2c$, where b, c are the exponents of v, w, so every $F \in S$ is congruent modulo I to a K-linear combination of standard monomials. There are three types of standard monomials:

$$\begin{array}{lll}
\text{no } x, \text{ no } y: & u^a v^b w^c z^k \ \mapsto\ t^k\, u^a\, v^b\, w^{c+k} & n \le r \\
\text{no } x, \text{ no } w,\ j > 0: & u^a v^b y^j z^k \ \mapsto\ t^{j+k}\, u^a\, v^{b+j}\, w^k & r < n \le q + r \\
\text{no } v, \text{ no } w,\ i > 0: & u^a x^i y^j z^k \ \mapsto\ t^{i+j+k}\, u^{a+i}\, v^j\, w^k & n > q + r
\end{array}$$

Write the image under φ as $t^n u^p v^q w^r$. The inequality on the right determines the type, and then the exponents can be read off from n, p, q, r. So φ sends distinct standard monomials to distinct monomials of $K[u, v, w, t]$, and is therefore injective on the K-span of the standard monomials.

Now let $F \in P$, and let $G \equiv F \bmod I$ be a K-linear combination of standard monomials. Then $\varphi(G) = \varphi(F) = 0$, so $G = 0$ and $F \in I$. Thus $I = P$ is prime, and $S/I \simeq K[u, v, w, tu, tv, tw]$ has fraction field $K(u, v, w, t)$, of transcendence degree 4 over K. So $\dim S/I = 4$ and $\operatorname{height}(I) = 6 - 4 = 2$ (Lecture 12). $\square$

Added after the lecture; not part of the projected slides.

On the blackboard: the Jacobian criterion on the rank-one locus

Claim. For every field K and every prime $\mathfrak{q} \neq \mathfrak{m} := (u, v, w, x, y, z)$ of $R := S/I$, the ring $R_{\mathfrak{q}}$ is regular.

Proof. View $\mathfrak{q}$ as a prime of S containing I, and let J be the Jacobian matrix of $f_1 := uy - vx$, $f_2 := uz - wx$, $f_3 := vz - wy$:

$$J = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{cccccc} \partial_u & \partial_v & \partial_w & \partial_x & \partial_y & \partial_z \end{array} \\ \begin{pmatrix} y & -x & 0 & -v & u & 0 \\ z & 0 & -x & -w & 0 & u \\ 0 & z & -y & 0 & -w & v \end{pmatrix} \end{array}$$

Some 2×2 minors of J (columns):

rows f_1, f_2:	u^2 (∂_y, ∂_z)	x^2 (∂_v, ∂_w)
rows f_1, f_3:	$-v^2$ (∂_x, ∂_z)	$-y^2$ (∂_u, ∂_w)
rows f_2, f_3:	w^2 (∂_x, ∂_y)	z^2 (∂_u, ∂_v)

We apply the Jacobian criterion with $\mathfrak{p} = I$ and $h = \operatorname{height}(I) = 2$: *if some* 2×2 *minor of* J *does not lie in* $\mathfrak{q}$, *then* $(S/I)_{\mathfrak{q}} = R_{\mathfrak{q}}$ *is regular.* The size of the minors is the height of I, not the number of generators: this is why rank 2 (not 3) is what matters. By the six minors on the right, a prime containing $I_2(J)$ contains u, v, w, x, y, z, so it is $\mathfrak{m}$. Hence $I_2(J) \not\subseteq \mathfrak{q}$, and $R_{\mathfrak{q}}$ is regular. □

Where is the rank? $I_2(J) \not\subseteq \mathfrak{q}$ says that J has rank ≥ 2 over $\kappa(\mathfrak{q})$; for $\mathfrak{q} = \mathfrak{m}_a$ with $a \in X := Z_K(I)$ this is $\operatorname{rank} J(a) \geq 2$, which holds for $a \neq 0$ by the six minors. In fact $\operatorname{rank} J(a) = 2$: the rows J_i of J satisfy

$$zJ_1 - yJ_2 + xJ_3 = (0, 0, 0, -f_3, f_2, -f_1), \qquad wJ_1 - vJ_2 + uJ_3 = (-f_3, f_2, -f_1, 0, 0, 0),$$

which vanish at a, and one of $(z, -y, x)$, $(w, -v, u)$ is nonzero at a. So $\dim T_a X = 6 - 2 = 4 = \dim R$ for $a \neq 0$, while $J(0) = 0$ and $\dim T_0 X = 6$. But ranks at points of X only see the maximal ideals $\mathfrak{m}_a$ (all of them only if $K = \overline{K}$); the condition $I_2(J) \not\subseteq \mathfrak{q}$ handles every prime, over every field.

Added after the lecture; not part of the projected slides.

Discrete valuation rings

Definition

A discrete valuation ring (DVR) is a regular local ring of dimension 1: a Noetherian local domain $(V, \mathfrak{m})$ with $\mathfrak{m} = (\pi) \neq 0$ principal.

- By Krull intersection every $0 \neq f \in V$ lies in $\mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$ for a unique n, so $f = \pi^n u$ with u a unit; put $\operatorname{ord}(f) := n$. Then $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$ and $\operatorname{ord}(f+g) \geq \min(\operatorname{ord} f, \operatorname{ord} g)$: ord is a discrete valuation. It extends to $\operatorname{Frac}(V)^\times \to \mathbb{Z}$, and $V = \{\alpha : \operatorname{ord}(\alpha) \geq 0\} \cup \{0\}$.
- The nonzero ideals of V are $\mathfrak{m}^n = (\pi^n)$: V is a PID with a unique nonzero prime.
- V is normal: if $\operatorname{ord}(\alpha) < 0$ then in $\alpha^n + c_1\alpha^{n-1} + \cdots + c_n$ $(c_i \in V)$ the first term has strictly smaller order than the rest, so the sum is nonzero.
- Examples: $K[x]_{(x)}$, $K[[x]]$, $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$; $K[x,y]_{(y-x^2)}$, where ord is the order of vanishing along the parabola; the local ring of a curve at any nonsingular point. Next lecture: a Noetherian normal local domain of dimension 1 is a DVR.

Regular rings

Theorem (Serre)

If $(R, \mathfrak{m})$ is a regular local ring and $\mathfrak{p} \in \operatorname{Spec} R$, then $R_{\mathfrak{p}}$ is regular.

All known proofs are homological. Auslander–Buchsbaum and Serre proved that a Noetherian local ring $(R, \mathfrak{m}, k)$ is regular iff k has a *finite free resolution*, an exact sequence $0 \to F_m \to \cdots \to F_0 \to k \to 0$ with each F_i f.g. free, iff every f.g. module has one. Localizing a finite free resolution of $R/\mathfrak{p}$ at $\mathfrak{p}$ gives one of $\kappa(\mathfrak{p})$ over $R_{\mathfrak{p}}$.

Definition

A Noetherian ring R is regular if $R_{\mathfrak{m}}$ is regular for every maximal ideal $\mathfrak{m}$, equivalently (by the theorem), for every prime.

- Fields, PIDs, and $K[x_1, \ldots, x_n]$ are regular; the lemma's proof shows that $R[x]$ is regular whenever R is, so $\mathbb{Z}[x_1, \ldots, x_n]$ is regular.
- For $K = \overline{K}$ the coordinate ring of a variety is regular iff the variety is nonsingular, e.g. the sphere $x^2 + y^2 + z^2 = 1$ ($\operatorname{char} K \neq 2$) but not the cusp $y^2 = x^3$.

Summary

- A Noetherian local ring $(R, \mathfrak{m}, k)$ is regular if $\dim R = \dim_k \mathfrak{m}/\mathfrak{m}^2$, i.e. if $\mathfrak{m}$ is generated by a system of parameters. Regular local rings are domains, and quotients by regular parameters (elements of $\mathfrak{m}$ independent mod $\mathfrak{m}^2$) are regular.
- For a variety $X \subseteq K^n$ $(K = \overline{K})$ with $I(X) = (f_1, \ldots, f_m)$ of height h and $a \in X$: the local ring of X at a is regular iff the tangent space $T_a X = \ker J(a)$ has dimension $\dim X$, iff the Jacobian matrix $J(a)$ has rank h. The Jacobian criterion ($I_h(J) \not\subseteq \mathfrak{q}$ implies $(S/\mathfrak{p})_{\mathfrak{q}}$ regular) holds over any field.
- Regular local rings of dimension 1 are discrete valuation rings: PIDs with a unique nonzero prime (π), with every nonzero element of the form $\pi^n u$. They are normal.
- Localizations of regular local rings are regular (Serre, via finite free resolutions), so "regular" is a local property. Polynomial rings over fields and over $\mathbb{Z}$ are regular.
- Next: normal rings and Serre's criterion (Jeffries §7.3).