18.705 Commutative Algebra
Lecture 14

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Recap, and where we are going

How many equations does it take to cut out an algebraic set in K^n?
Over $K = \mathbb{R}$ one always suffices: $f_1 = \cdots = f_m = 0$ iff $f_1^2 + \cdots + f_m^2 = 0$.

Over $K = \overline{K}$ the Nullstellensatz makes this a question about rings: is there an N, depending only on n, such that every ideal of $K[x_1, \ldots, x_n]$ has the same radical as an ideal generated by at most N elements? Today: heights versus numbers of generators.

- $\operatorname{height}(\mathfrak{p})$ is the supremum of the lengths of chains of primes ending at $\mathfrak{p}$, $\operatorname{height}(I) := \min_{\mathfrak{p} \in \operatorname{Min}(I)} \operatorname{height}(\mathfrak{p})$, and $\dim R_{\mathfrak{p}} = \operatorname{height}(\mathfrak{p})$ (Lecture 11).
  In an affine domain of dimension d, $\operatorname{height}(\mathfrak{p}) + \dim R/\mathfrak{p} = d$ (Lecture 12).
- A Noetherian ring of dimension zero is Artinian (Lecture 13).
- Krull intersection: if $(R, \mathfrak{m})$ is Noetherian local then $\bigcap_n \mathfrak{m}^n = 0$ (Lecture 10).
- Prime avoidance: An ideal lying in a finite union of primes lies in one of them (Lecture 10).
- The symbolic power $\mathfrak{p}^{(n)} := \mathfrak{p}^n R_{\mathfrak{p}} \cap R$ of a prime $\mathfrak{p}$ is $\mathfrak{p}$-primary and contains $\mathfrak{p}^n$.

Krull's principal ideal theorem

Theorem (Krull's principal ideal theorem)

Let R be Noetherian and $f \in R$. Every minimal prime of (f) has height at most 1.
Proof: To the blackboard!

- This is stronger than $\mathrm{height}(f) \leq 1$, which only says that *some* minimal prime of (f) has height at most 1.
- If R is a domain and $f \neq 0$ is not a unit, every minimal prime of (f) has height exactly 1. In a UFD these are the ideals (p) for the prime factors p of f.
- Geometrically: if $X \subseteq K^n$ $(K = \overline{K})$ is an irreducible variety of dimension d and f vanishes on a nonempty proper subset of X, then every irreducible component of $X \cap Z_K(f)$ has dimension $d-1$: for $R = K[x_1, \ldots, x_n]/I(X)$ the components are the $Z_K(\mathfrak{q})$ with $\mathfrak{q} \in \mathrm{Min}(fR)$, and $\dim R/\mathfrak{q} = d - \mathrm{height}(\mathfrak{q}) = d - 1$.
- The Noetherian hypothesis cannot be dropped.

On the blackboard: Krull's principal ideal theorem

Theorem. Let R be a Noetherian ring and $f \in R$. Every minimal prime of (f) has height at most 1.

Proof. Suppose $\mathfrak{p}$ is minimal over (f) and $\operatorname{height}(\mathfrak{p}) \geq 2$, so there are primes $\mathfrak{p}_0 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$.
The primes of $R_{\mathfrak{p}}/\mathfrak{p}_0 R_{\mathfrak{p}}$ correspond to primes $\mathfrak{p}'$ of R with $\mathfrak{p}_0 \subseteq \mathfrak{p}' \subseteq \mathfrak{p}$; the only one containing f is $\mathfrak{p}$. Replacing R by $R_{\mathfrak{p}}/\mathfrak{p}_0 R_{\mathfrak{p}}$ we may thus assume that $(R, \mathfrak{m})$ is a Noetherian local domain, that $\mathfrak{m}$ is the only prime containing f, and that $0 \neq \mathfrak{q} \subsetneq \mathfrak{m}$ is a prime. Then $\bar{R} := R/(f)$ is Noetherian with a single prime ideal, hence Artinian (Lecture 13).

For $n \geq 1$ the symbolic power $\mathfrak{q}^{(n)} = \mathfrak{q}^n R_{\mathfrak{q}} \cap R$ is $\mathfrak{q}$-primary and contains $\mathfrak{q}^n$. The descending chain $\mathfrak{q}^{(1)}\bar{R} \supseteq \mathfrak{q}^{(2)}\bar{R} \supseteq \cdots$ of ideals of $\bar{R}$ stabilizes: there is an n with $\mathfrak{q}^{(n)}\bar{R} = \mathfrak{q}^{(m)}\bar{R}$, i.e. $\mathfrak{q}^{(n)} \subseteq \mathfrak{q}^{(m)} + (f)$, for all $m \geq n$.

Claim: $\mathfrak{q}^{(n)} = \mathfrak{q}^{(m)}$ *for all* $m \geq n$. Given $x \in \mathfrak{q}^{(n)}$, write $x = y + fr$ with $y \in \mathfrak{q}^{(m)} \subseteq \mathfrak{q}^{(n)}$ and $r \in R$. Then $fr = x - y \in \mathfrak{q}^{(n)}$ and $f \notin \mathfrak{q} = \sqrt{\mathfrak{q}^{(n)}}$ (as $\mathfrak{q} \neq \mathfrak{m}$), so $r \in \mathfrak{q}^{(n)}$ because $\mathfrak{q}^{(n)}$ is primary. Thus $\mathfrak{q}^{(n)} = \mathfrak{q}^{(m)} + f\mathfrak{q}^{(n)} \subseteq \mathfrak{q}^{(m)} + m\mathfrak{q}^{(n)}$, so $\mathfrak{q}^{(n)}/\mathfrak{q}^{(m)} = \mathfrak{m}\,(\mathfrak{q}^{(n)}/\mathfrak{q}^{(m)})$, and NAK gives $\mathfrak{q}^{(n)} = \mathfrak{q}^{(m)}$.

Now $R \subseteq R_{\mathfrak{q}}$ since R is a domain, and $\mathfrak{q}^{(m)} \subseteq \mathfrak{q}^m R_{\mathfrak{q}}$, so $\mathfrak{q}^{(n)} = \bigcap_{m \geq n} \mathfrak{q}^{(m)} \subseteq \bigcap_m \mathfrak{q}^m R_{\mathfrak{q}} = 0$ by the Krull intersection theorem for the Noetherian local ring $(R_{\mathfrak{q}}, \mathfrak{q}R_{\mathfrak{q}})$.
But $\mathfrak{q} \neq 0$: for $0 \neq a \in \mathfrak{q}$ we have $0 \neq a^n \in \mathfrak{q}^n \subseteq \mathfrak{q}^{(n)}$, a contradiction. □

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Krull's height theorem

Lemma

Let R be Noetherian, $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ primes, and $f \in \mathfrak{p}_2$. Then there is a prime $\mathfrak{p}_1'$ with $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1' \subsetneq \mathfrak{p}_2$ and $f \in \mathfrak{p}_1'$.

Theorem (Krull's height theorem)

Let R be a Noetherian ring and $I = (f_1, \ldots, f_n)$ an ideal generated by n elements. Then every minimal prime of I has height at most n.

Proof: To the blackboard!

- So $\operatorname{height}(I) \leq \mu(I)$, the minimal number of generators of I. For a Noetherian local ring $(R, \mathfrak{m}, k)$ this gives $\dim R = \operatorname{height}(\mathfrak{m}) \leq \mu(\mathfrak{m}) = \dim_k \mathfrak{m}/\mathfrak{m}^2 < \infty$ (NAK, Lecture 5): *Noetherian local rings have finite dimension*.
- Geometrically: if $X \subseteq K^n$ $(K = \overline{K})$ is an irreducible variety of dimension d, then every irreducible component of $X \cap Z_K(f_1, \ldots, f_r)$ has dimension at least $d - r$. Each equation cuts the dimension by at most one.

On the blackboard: Krull's height theorem

Lemma. Let R be Noetherian, $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ primes, and $f \in \mathfrak{p}_2$. Then there is a prime $\mathfrak{p}_1'$ with $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1' \subsetneq \mathfrak{p}_2$ and $f \in \mathfrak{p}_1'$.

Proof. If $f \in \mathfrak{p}_0$ take $\mathfrak{p}_1' := \mathfrak{p}_1$. Otherwise, $R' := R_{\mathfrak{p}_2}/\mathfrak{p}_0 R_{\mathfrak{p}_2}$ is a Noetherian local domain of dimension ≥ 2 in which the image of f is a nonzero nonunit. Let q' be a minimal prime of fR'. Then $q' \neq 0$, since R' is a domain and $f \neq 0$, and q' is not the maximal ideal of R', since $\operatorname{height}(q') \leq 1 < \dim R'$ by the principal ideal theorem. The preimage $\mathfrak{p}_1'$ of q' in R satisfies $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1' \subsetneq \mathfrak{p}_2$ and $f \in \mathfrak{p}_1'$. □

Theorem. Let R Noetherian and $I = (f_1, \ldots, f_n)$. Every minimal prime of I has height at most n.

Proof. Induction on n; the case $n = 1$ is the principal ideal theorem. Let $n \geq 2$, let $\mathfrak{p}$ be a minimal prime of I, and let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_h = \mathfrak{p}$ be a chain of primes; we must show $h \leq n$.

We may assume $f_1 \in \mathfrak{p}_1$. Let i be the least index with $f_1 \in \mathfrak{p}_i$. If $i \geq 2$, the lemma applied to $\mathfrak{p}_{i-2} \subsetneq \mathfrak{p}_{i-1} \subsetneq \mathfrak{p}_i$ yields a prime $\mathfrak{p}_{i-1}' \ni f_1$ with $\mathfrak{p}_{i-2} \subsetneq \mathfrak{p}_{i-1}' \subsetneq \mathfrak{p}_i$; replacing $\mathfrak{p}_{i-1}$ by $\mathfrak{p}_{i-1}'$ gives a chain of the same length in which the least index is $i - 1$. Repeat until $i \leq 1$.

Induction step. In $\bar{R} := R/(f_1)$ the primes containing $(\bar{f}_2, \ldots, \bar{f}_n)$ correspond to the primes of R containing I, so $\bar{\mathfrak{p}}$ is a minimal prime of the ideal $(\bar{f}_2, \ldots, \bar{f}_n)$, and $\operatorname{height}(\bar{\mathfrak{p}}) \leq n - 1$ by the inductive hypothesis. Since $f_1 \in \mathfrak{p}_1$, the chain $\bar{\mathfrak{p}}_1 \subsetneq \cdots \subsetneq \bar{\mathfrak{p}}_h = \bar{\mathfrak{p}}$ has length $h - 1$, so $h - 1 \leq n - 1$. □

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Examples

- The bound is sharp: $(x_1, \ldots, x_n) \subseteq K[x_1, \ldots, x_n]$ has height n. An ideal of height n generated by n elements is a complete intersection; for example $(y - x^2,\ z - x^3) \subseteq K[x, y, z]$ has height $3 - \dim K[x, y, z]/(y - x^2, z - x^3) = 3 - \dim K[x] = 2$.

- The minimal primes of $I = (xy, xz) = (x) \cap (y, z)$ in $K[x, y, z]$ have heights 1 and 2: $Z_K(I)$ is the union of a plane and a line.

- Noetherian is necessary. Let $R := K[x, xy, xy^2, \ldots] \subseteq K[x, y]$ and $\mathfrak{m} := (x, xy, xy^2, \ldots) = xK[x, y]$, a maximal ideal ($R/\mathfrak{m} = K$). As $(xy^a)^2 = x \cdot xy^{2a} \in (x)$ we have $\sqrt{(x)} = \mathfrak{m}$, so $\mathfrak{m}$ is the unique minimal prime of (x). But $\mathfrak{p} := yK[x, y] \cap R = (xy, xy^2, \ldots)$ is prime with $0 \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$, so $\operatorname{height}(\mathfrak{m}) \geq 2$. (Here $\mathfrak{m}$ is not finitely generated.)

Heights and numbers of generators

Theorem

Let R be a Noetherian ring.

1. *Every prime of height h is a minimal prime of an ideal generated by h elements.*
2. *If $\dim R = d < \infty$ then for every ideal I there are $f_1, \ldots, f_{d+1} \in I$ with $\sqrt{I} = \sqrt{(f_1, \ldots, f_{d+1})}$. If R is local and $I \neq R$, then d elements suffice.*

Proof: To the blackboard!

Corollary

Every algebraic subset of K^n is the zero locus of $n+1$ polynomials.

- In fact n polynomials suffice (Eisenbud–Evans, Storch), and for homogeneous ideals of $K[x_1, \ldots, x_n]$ the proof below gives n via homogeneous prime avoidance.
- The bound concerns $\sqrt{I}$, not I: the ideal $(x, y)^n \subseteq K[x, y]$ needs $n+1$ generators (check!), but it has the same radical as (x^n, y^n).

On the blackboard: heights and numbers of generators

Setup. Let R be Noetherian and I an ideal. Choose $f_1, f_2, \ldots \in I$ successively so that, with $J_i := (f_1, \ldots, f_i)$, each f_i lies outside every $\mathfrak{q} \in \operatorname{Min}(J_{i-1}) \setminus V(I)$. Since $\operatorname{Min}(J_{i-1})$ is finite (Lecture 9), prime avoidance makes this possible as long as $I \not\subseteq \mathfrak{q}$ for these $\mathfrak{q}$, which holds by definition.

Lemma. Every minimal prime of J_i either contains I or has height i.

Proof. Induct on i: minimal primes of $J_0 = (0)$ have height 0. For $i \geq 1$, a minimal prime $\mathfrak{q}$ of J_i contains some $\mathfrak{p} \in \operatorname{Min}(J_{i-1})$. If $\mathfrak{p} \supseteq I$ then $\mathfrak{q} \supseteq I$. Otherwise $\operatorname{height}(\mathfrak{p}) = i - 1$ and $f_i \in \mathfrak{q} \setminus \mathfrak{p}$, so $\mathfrak{q} \supsetneq \mathfrak{p}$ and $\operatorname{height}(\mathfrak{q}) \geq i$, while $\operatorname{height}(\mathfrak{q}) \leq i$ by Krull. □

Theorem. (1) A prime $\mathfrak{p}$ of height h is minimal over an ideal generated by h elements of $\mathfrak{p}$. (2) If $\dim R = d$ then every ideal I satisfies $\sqrt{I} = \sqrt{(f_1, \ldots, f_{d+1})}$ for some $f_i \in I$, and d elements suffice if R is local and $I \neq R$.

Proof. (1) Run the setup with $I = \mathfrak{p}$. For $i \leq h$ every $\mathfrak{q} \in \operatorname{Min}(J_{i-1})$ has height $\leq i - 1 < h$ by Krull, so $\mathfrak{q} \not\supseteq \mathfrak{p}$; thus the recipe produces $f_1, \ldots, f_h \in \mathfrak{p}$. As $J_h \subseteq \mathfrak{p}$, some $\mathfrak{q} \in \operatorname{Min}(J_h)$ lies in $\mathfrak{p}$; by the lemma either $\mathfrak{q} \supseteq \mathfrak{p}$, or $\operatorname{height}(\mathfrak{q}) = h = \operatorname{height}(\mathfrak{p})$, and in both cases $\mathfrak{q} = \mathfrak{p}$. So $\mathfrak{p} \in \operatorname{Min}(J_h)$.

(2) Run the setup with I. If at some step $i \leq d + 1$ every minimal prime of J_{i-1} contains I, then $V(J_{i-1}) = \bigcup_{\mathfrak{q} \in \operatorname{Min}(J_{i-1})} V(\mathfrak{q}) \subseteq V(I) \subseteq V(J_{i-1})$, so $\sqrt{I} = \sqrt{J_{i-1}}$ and $i - 1 \leq d$ elements suffice. Otherwise we reach J_{d+1}, whose minimal primes contain I or have height $d + 1 > \dim R$; the latter is impossible, so $\sqrt{I} = \sqrt{J_{d+1}}$ as before. If $(R, \mathfrak{m})$ is local and $I \neq R$, the only prime of height d is $\mathfrak{m}$, which contains I, so every minimal prime of J_d already contains I and $\sqrt{I} = \sqrt{J_d}$. □

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Systems of parameters

Corollary

For a Noetherian local ring $(R, \mathfrak{m}, k)$, $\dim R$ *is the least* n *such that* $\sqrt{(f_1, \ldots, f_n)} = \mathfrak{m}$ *for some* $f_1, \ldots, f_n \in \mathfrak{m}$*; in particular* $\dim R \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

Definition

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. A system of parameters of R is a sequence $x_1, \ldots, x_d \in \mathfrak{m}$ with $\sqrt{(x_1, \ldots, x_d)} = \mathfrak{m}$, i.e. with $R/(x_1, \ldots, x_d)$ Artinian. Elements $x_1, \ldots, x_m \in \mathfrak{m}$ are parameters if they are part of a system of parameters.

Lemma

For $x_1, \ldots, x_m \in \mathfrak{m}$, $\dim R/(x_1, \ldots, x_m) \geq \dim R - m$*, with equality iff the* x_i *are parameters.* **Proof**: To the blackboard!

- x, y and $x^2 + y^3$, y^5 are systems of parameters of $K[x, y]_{(x,y)}$, but x, xy is not.
- $x \in \mathfrak{m}$ is a parameter iff it lies in no minimal prime $\mathfrak{p}$ with $\dim R/\mathfrak{p} = \dim R$ (check!); in $K[x, y]_{(x,y)}/(xy)$, $x + y$ is a parameter but x is not.

On the blackboard: systems of parameters

Corollary. For a Noetherian local ring $(R, \mathfrak{m}, k)$, $\dim R$ is the least n for which there exist $f_1, \dots, f_n \in \mathfrak{m}$ with $\sqrt{(f_1, \dots, f_n)} = \mathfrak{m}$; in particular, $\dim R \le \dim_k \mathfrak{m}/\mathfrak{m}^2$.

Proof. If $\sqrt{(f_1, \dots, f_n)} = \mathfrak{m}$ then $\mathfrak{m}$ is the only prime containing $(f_1, \dots, f_n)$, hence a minimal prime of it, so $\dim R = \operatorname{height}(\mathfrak{m}) \le n$ by Krull's height theorem; and part (2) of the theorem with $I = \mathfrak{m}$ produces $n = \dim R$ such elements. Finally, $\mathfrak{m}$ is generated by $\dim_k \mathfrak{m}/\mathfrak{m}^2$ elements (NAK, Lecture 5). $\square$

Lemma. For $x_1, \dots, x_m \in \mathfrak{m}$ we have $\dim R/(x_1, \dots, x_m) \ge \dim R - m$, with equality iff $x_1, \dots, x_m$ are parameters.

Proof. Let $d := \dim R$, $\bar{R} := R/(x_1, \dots, x_m)$, and $s := \dim \bar{R}$. Lift a system of parameters of $\bar{R}$ to $y_1, \dots, y_s \in \mathfrak{m}$. The primes of R containing $(x_1, \dots, x_m, y_1, \dots, y_s)$ correspond to the primes of $\bar{R}$ containing $(\bar{y}_1, \dots, \bar{y}_s)$, of which $\bar{\mathfrak{m}}$ is the only one, so $\sqrt{(x_1, \dots, x_m, y_1, \dots, y_s)} = \mathfrak{m}$ and $m + s \ge d$ by the corollary.

If $m + s = d$ then $x_1, \dots, x_m, y_1, \dots, y_s$ is a system of parameters, so $x_1, \dots, x_m$ are parameters. Conversely, if $x_1, \dots, x_m$ extend to a system of parameters $x_1, \dots, x_d$, then by the same correspondence $\sqrt{(\bar{x}_{m+1}, \dots, \bar{x}_d)} = \bar{\mathfrak{m}}$ in $\bar{R}$, so $s \le d - m$ by the corollary applied to $\bar{R}$, and hence $s = d - m$. $\square$

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Summary

- Krull's principal ideal theorem: in a Noetherian ring, every minimal prime of a principal ideal has height at most 1.
- Krull's height theorem: every minimal prime of an ideal generated by n elements has height $\leq n$. Geometrically, each equation cuts the dimension by at most one.
- Conversely, a prime of height h is minimal over h elements, and in a Noetherian ring of dimension d every ideal has the same radical as an ideal generated by $d+1$ elements (d for proper ideals of a local ring). Every algebraic subset of K^n is cut out by $n+1$ equations.
- A Noetherian local ring $(R, \mathfrak{m}, k)$ has $\dim R \leq \dim_k \mathfrak{m}/\mathfrak{m}^2 < \infty$, and $\dim R$ is the length of a system of parameters: elements $x_1, \ldots, x_d \in \mathfrak{m}$ with $\sqrt{(x_1, \ldots, x_d)} = \mathfrak{m}$.
- Next: regular local rings, those with $\dim R = \dim_k \mathfrak{m}/\mathfrak{m}^2$ (Jeffries §7.2).