18.705 Commutative Algebra
Lecture 13

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Recap, and where we are going

Our next goal is Krull's height theorem: in a Noetherian ring, a prime minimal over an ideal generated by n elements has height at most n. Its proof reduces to Noetherian rings of dimension 0, which we study today. Some facts worth recalling:

- R is Noetherian iff every nonempty set of ideals has a maximal element; a Noetherian ring has finitely many minimal primes (Lecture 9), and its nilradical is nilpotent (generated by finitely many nilpotents).
- $\mathfrak{q} \subsetneq R$ is primary if $xy \in \mathfrak{q}$ implies $x \in \mathfrak{q}$ or $y \in \sqrt{\mathfrak{q}}$, i.e. if every zerodivisor of $R/\mathfrak{q}$ is nilpotent; then $\mathfrak{p} := \sqrt{\mathfrak{q}}$ is prime and $\mathfrak{q}$ is $\mathfrak{p}$-primary (Lecture 10).
- NAK for a local ring $(R, \mathfrak{m})$: if M is finitely generated and $\mathfrak{m}M = M$ then $M = 0$ (Lecture 5).
- CRT: if $I_1, \ldots, I_n$ are pairwise comaximal ($I_i + I_j = R$ for $i \neq j$), then $I_1 \cap \cdots \cap I_n = I_1 \cdots I_n$ and $R/(I_1 \cap \cdots \cap I_n) \simeq R/I_1 \times \cdots \times R/I_n$.

Ideals supported at finitely many closed points

Theorem

Let I be an ideal of a ring R with $V(I) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ a finite set of maximal ideals. Then $\mathfrak{q}_i := IR_{\mathfrak{m}_i} \cap R$ is $\mathfrak{m}_i$-primary and

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = \mathfrak{q}_1 \cdots \mathfrak{q}_n, \qquad R/I \simeq R/\mathfrak{q}_1 \times \cdots \times R/\mathfrak{q}_n.$$

Proof: To the blackboard!

- Example: $K[x]/(f) \simeq \prod_i K[x]/(p_i^{e_i})$ for $f = \prod_i p_i^{e_i}$ with distinct irreducible p_i: functions on a finite set of points form the product of the local rings at the points.
- We will apply this with $I = 0$: a ring with finitely many primes, all maximal, is a finite product of local rings.

On the blackboard: ideals supported at finitely many closed points

Theorem. Let I be an ideal of a ring R with $V(I) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ a finite set of maximal ideals. Then $q_i := IR_{\mathfrak{m}_i} \cap R$ is $\mathfrak{m}_i$-primary and $I = q_1 \cap \cdots \cap q_n = q_1 \cdots q_n$, so $R/I \simeq R/q_1 \times \cdots \times R/q_n$.

Proof. Recall that for a multiplicative set W and an ideal J, the *saturation* of J is the ideal $JW^{-1}R \cap R = \{r \in R : wr \in J \text{ for some } w \in W\}$, and $(JW^{-1}R \cap R)W^{-1}R = JW^{-1}R$, as both sides contain J and the saturation lies in $JW^{-1}R$. Ideals $J \subseteq J'$ with $JR_{\mathfrak{m}} = J'R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ are equal.

q_i *is* $\mathfrak{m}_i$*-primary:* the primes of $R_{\mathfrak{m}_i}$ containing $IR_{\mathfrak{m}_i}$ correspond to the primes $\mathfrak{p}$ with $I \subseteq \mathfrak{p} \subseteq \mathfrak{m}_i$, and $\mathfrak{m}_i$ is the only one. The radical of an ideal is the intersection of the primes containing it, so $\sqrt{IR_{\mathfrak{m}_i}} = \mathfrak{m}_i R_{\mathfrak{m}_i}$ is maximal and $IR_{\mathfrak{m}_i}$ is $\mathfrak{m}_i R_{\mathfrak{m}_i}$-primary (see Lecture 10). Contractions of primary ideals are primary: if $xy \in q_i$ and $x \notin q_i$ then $x \notin IR_{\mathfrak{m}_i}$, so $y^k \in IR_{\mathfrak{m}_i} \cap R = q_i$ for some k. And $\sqrt{q_i} = \sqrt{IR_{\mathfrak{m}_i}} \cap R = \mathfrak{m}_i R_{\mathfrak{m}_i} \cap R = \mathfrak{m}_i$.

$I = \bigcap_i q_i$*:* clearly $I \subseteq \bigcap_i q_i$, so it suffices to show $(\bigcap_i q_i)R_{\mathfrak{m}} \subseteq IR_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$. If $\mathfrak{m} \notin V(I)$ then $IR_{\mathfrak{m}} = R_{\mathfrak{m}}$. If $\mathfrak{m} = \mathfrak{m}_j$ then $(\bigcap_i q_i)R_{\mathfrak{m}_j} \subseteq q_j R_{\mathfrak{m}_j} = (IR_{\mathfrak{m}_j} \cap R)R_{\mathfrak{m}_j} = IR_{\mathfrak{m}_j}$.

Comaximality: for $i \neq j$, $V(q_i + q_j) = V(q_i) \cap V(q_j) = V(\mathfrak{m}_i) \cap V(\mathfrak{m}_j) = \emptyset$, so $q_i + q_j = R$, and the Chinese remainder theorem gives $\bigcap_i q_i = \prod_i q_i$ and $R/I \simeq \prod_i R/q_i$. □

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Simple modules and length

Definition

A nonzero module is simple if it has no nonzero proper submodules. A composition series of M is a chain $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ of submodules with every M_i/M_{i-1} simple. M has finite length if it has a composition series, and its length $\ell(M) = \ell_R(M)$ is the minimal length n of a composition series (∞ if none exist).

- M is simple iff $M \simeq R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$: a simple M is generated by any $0 \neq m \in M$, so $M \simeq R/\operatorname{ann}(m)$, and R/I is simple iff I is maximal, as its submodules are the J/I with $I \subseteq J$.
- So a composition series has $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ as in a prime filtration (Lecture 9), but the simplicity of the quotients forces all the $\mathfrak{p}_i$ to be maximal ideals.
- If $\mathfrak{m}M = 0$ for a maximal ideal $\mathfrak{m}$, then $\ell_R(M) = \dim_{R/\mathfrak{m}} M$: the submodules of M are its $R/\mathfrak{m}$-subspaces, and a composition series is a full flag.
- $\ell_{\mathbb{Z}}(\mathbb{Z}/(p^n)) = n$ via $0 \subsetneq (p^{n-1})/(p^n) \subsetneq \cdots \subsetneq \mathbb{Z}/(p^n)$, and $\ell_{\mathbb{Z}}(\mathbb{Z}/(n))$ is the number of prime factors of n with multiplicity; $\mathbb{Z}$ itself does not have finite length.

Artinian rings and modules

Definition

A module M is Artinian if every descending chain $M_1 \supseteq M_2 \supseteq \cdots$ of submodules stabilizes; equivalently, every nonempty set of submodules of M has a minimal element. A ring R is Artinian if it is Artinian as an R-module.

- As for Noetherian modules: in an exact sequence $0 \to L \to M \to N \to 0$, M is Artinian iff L and N are; so quotients, submodules, and finite direct sums of Artinian modules are Artinian.
- A vector space is Artinian iff it is Noetherian iff it is finite-dimensional.
- Fields, finite rings, $\mathbb{Z}/(n)$, $K[x]/(f)$, and finite-dimensional K-algebras are Artinian; $\mathbb{Z}$ and $K[x]$ are not: $(x) \supsetneq (x^2) \supsetneq (x^3) \supsetneq \cdots$.
- Artinian rings turn out to be Noetherian, but Artinian modules need not be: for $R := \mathbb{C}[[x]]$, the module $M := R[1/x]/R$ of "polar tails" of Laurent series is Artinian but not finitely generated: its proper submodules are exactly the $R \cdot x^{-n}$, of length n (check!).

The Jordan–Hölder theorem

Theorem (Jordan–Hölder)

Let M have finite length. (i) If $N \subsetneq M$ then $\ell(N) < \ell(M)$. (ii) Every chain of submodules of M has length $\leq \ell(M)$. (iii) Every composition series of M has length $\ell(M)$. (iv) Every chain of submodules of M refines to a composition series.
Proof: To the blackboard!

Corollary

1. *If $0 \to L \to M \to N \to 0$ is exact then $\ell(M) = \ell(L) + \ell(N)$.*
2. *M has finite length iff M is Noetherian and Artinian.*

Proof: (1) Intersect a composition series of M with L, or map it to M/L, and delete repetitions; conversely, splice composition series of L and N and use (iii). (2) $\Rightarrow$: (ii). $\Leftarrow$: pick a minimal nonzero submodule M_1 (DCC), then $M_2 \supsetneq M_1$ with M_2/M_1 minimal nonzero in M/M_1, etc.; ACC forces this to end, at M.

On the blackboard: the Jordan–Hölder theorem

Theorem. Let M have finite length. (i) If $N \subsetneq M$ then $\ell(N) < \ell(M)$. (ii) Every chain of submodules of M has length $\leq \ell(M)$. (iii) Every composition series of M has length $\ell(M)$. (iv) Every chain of submodules of M can be refined to a composition series.

Proof. (i) Let $0 = M_0 \subsetneq \cdots \subsetneq M_n = M$ be a composition series with $n = \ell(M)$, and put $N_i := N \cap M_i$. The natural map $N_i/N_{i-1} \to M_i/M_{i-1}$ is injective, since $N_i \cap M_{i-1} = N_{i-1}$, so each N_i/N_{i-1} is zero or simple, and deleting repetitions from $0 = N_0 \subseteq \cdots \subseteq N_n = N$ gives a composition series of N of length $\leq n$; thus $\ell(N) \leq n$. If $\ell(N) = n$ there are no repetitions, so every $N_i/N_{i-1} \to M_i/M_{i-1}$ is an injective map of simple modules, hence an isomorphism; then $N_{i-1} = M_{i-1}$ implies $N_i = M_i$, and by induction $N = N_n = M_n = M$, a contradiction.

(ii) For a chain $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_k = M$, applying (i) to $N_{i-1} \subsetneq N_i$ gives $0 = \ell(N_0) < \ell(N_1) < \cdots < \ell(N_k) = \ell(M)$, so $k \leq \ell(M)$.

(iii) A composition series has length $\geq \ell(M)$ by definition and $\leq \ell(M)$ by (ii).

(iv) If a chain is not a composition series, some quotient N_i/N_{i-1} is zero or not simple, so we can delete a term or insert a submodule strictly between N_{i-1} and N_i. By (ii) the length cannot exceed $\ell(M)$, so after finitely many steps we reach a composition series. $\square$

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Artinian rings are Noetherian of dimension zero

Theorem

For a nonzero ring R the following are equivalent:

1. *R is Noetherian and $\dim R = 0$;*
2. *R is a finite product of Noetherian local rings of dimension 0;*
3. *R has finite length as an R-module;*
4. *R is Artinian.*

Proof: (1)⇒(2): every prime is minimal and maximal, so $\operatorname{Spec} R = \operatorname{Min}(R)$ is finite, and the theorem above with $I = 0$ gives $R \simeq \prod_i R/\mathfrak{q}_i$ with each $R/\mathfrak{q}_i$ local, Noetherian, and of dimension 0. (2)⇒(3): lengths add over finite products, so let $(R, \mathfrak{m})$ be Noetherian local of dimension 0; then $\mathfrak{m} = \sqrt{0}$ is nilpotent, say $\mathfrak{m}^N = 0$, and $\ell(R) = \sum_{i<N} \ell(\mathfrak{m}^i/\mathfrak{m}^{i+1}) < \infty$, as each $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finitely generated $R/\mathfrak{m}$-vector space. (3)⇒(4): Jordan–Hölder, corollary (2). (4)⇒(1): to the blackboard!

In particular, an Artinian ring has finitely many primes, all maximal, and its finitely generated modules have finite length; an Artinian local ring $(R, \mathfrak{m})$ has $\mathfrak{m}$ nilpotent.

On the blackboard: Artinian rings are Noetherian

Claim. An Artinian ring R is Noetherian of dimension 0.

Proof. *Every prime is maximal:* for $\mathfrak{p}$ prime, $R/\mathfrak{p}$ is an Artinian domain; for $0 \neq a \in R/\mathfrak{p}$ the chain $(a) \supseteq (a^2) \supseteq \cdots$ stabilizes, so $a^n = a^{n+1}b$ for some b, and cancelling a^n gives $ab = 1$. So $R/\mathfrak{p}$ is a field and $\mathfrak{p}$ is maximal.

Finitely many maximal ideals: if $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$ are distinct maximal ideals, the chain $\mathfrak{m}_1 \supseteq \mathfrak{m}_1 \cap \mathfrak{m}_2 \supseteq \cdots$ stabilizes, so $\mathfrak{m}_{n+1} \supseteq \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n$ for some n; the prime $\mathfrak{m}_{n+1}$ then contains some $\mathfrak{m}_i$, hence equals it, a contradiction.

Reduction to the local case: now $V(0) = \operatorname{Spec} R = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$, so $R \simeq \prod_i R/q_i$ with each R/q_i local, and Artinian as a quotient of R. A finite product of Noetherian rings is Noetherian, so we may assume $(R, \mathfrak{m})$ is Artinian local.

$\mathfrak{m}$ *is nilpotent:* the chain $\mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots$ stabilizes, say $\mathfrak{m}^n = \mathfrak{m}^{n+1} =: J$, so $\mathfrak{m}J = J$. Suppose $J \neq 0$. The set of ideals I with $IJ \neq 0$ contains $\mathfrak{m}$, so it has a minimal element I_0. Some $x \in I_0$ has $xJ \neq 0$, so $I_0 = (x)$ by minimality. Then $(x\mathfrak{m})J = xJ \neq 0$ and $x\mathfrak{m} \subseteq (x)$, so $x\mathfrak{m} = (x)$ by minimality, and NAK gives $(x) = 0$, contradicting $xJ \neq 0$. Thus $\mathfrak{m}^n = 0$.

Finite length: in $0 = \mathfrak{m}^n \subseteq \mathfrak{m}^{n-1} \subseteq \cdots \subseteq \mathfrak{m} \subseteq R$ each $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is an Artinian R-module killed by $\mathfrak{m}$, i.e. an $R/\mathfrak{m}$-vector space with DCC on subspaces, hence finite-dimensional and of finite length. So $\ell(R) < \infty$, and R is Noetherian by Jordan–Hölder corollary (2). $\square$

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Modules of finite length

Recall (Lecture 9): $\mathrm{Supp}(M) := \{\mathfrak{p} : M_{\mathfrak{p}} \neq 0\}$, and $\mathrm{Ass}(M)$ is the set of primes of the form $\mathrm{ann}_R(m)$ with $m \in M$. For R Noetherian and M finitely generated, $\mathrm{Supp}(M) = V(\mathrm{ann}\, M)$ and $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$ contains its minimal primes.

Proposition

Let R be Noetherian and M a finitely generated R-module. The following are equivalent: (1) $\ell(M) < \infty$; (2) every $\mathfrak{p} \in \mathrm{Ass}(M)$ is maximal; (3) every $\mathfrak{p} \in \mathrm{Supp}(M)$ is maximal; (4) $R/\mathrm{ann}(M)$ is Artinian.

Proof: (1)⇒(2): a composition series is a prime filtration with maximal primes, and $\mathrm{Ass}(M)$ lies in the set of primes of any prime filtration (Lecture 9). (2)⇒(3): the minimal primes of $\mathrm{Supp}(M)$ lie in $\mathrm{Ass}(M)$. (3)⇒(4): $R/\mathrm{ann}(M)$ is Noetherian with $\mathrm{Spec}(R/\mathrm{ann}\, M) = \mathrm{Supp}(M)$ of dimension 0. (4)⇒(1): M is a quotient of $(R/\mathrm{ann}\, M)^n$ for some n, which has finite length.

For $(R, \mathfrak{m})$ Noetherian local and M finitely generated: $\ell(M) < \infty$ iff $\mathrm{Supp}(M) \subseteq \{\mathfrak{m}\}$ iff $\mathfrak{m}^n M = 0$ for some n; in particular, for $I \neq R$, $\ell(R/I) < \infty$ iff I is $\mathfrak{m}$-primary.

Coefficient fields and intersection multiplicities

Definition

A coefficient field of a local ring $(R, \mathfrak{m}, k)$ is a subfield $K \subseteq R$ for which $K \to R \to R/\mathfrak{m} = k$ is an isomorphism.

- If R has a coefficient field K, every finite length R-module M is a K-vector space with $\ell_R(M) = \dim_K M$: in a composition series each $M_i/M_{i-1} \simeq k \simeq K$ is one-dimensional over K.
- For $K = \overline{K}$ and $a \in K^n$, the local ring $K[x_1, \ldots, x_n]_{\mathfrak{m}_a}/I$ of a variety at a has coefficient field K. So for $f, g \in K[x, y]$ and $R := K[x, y]_{\mathfrak{m}_a}$, $\ell(R/(f, g)) = \dim_K R/(f, g)$ is the intersection multiplicity of the curves $f = 0$ and $g = 0$ at a: the parabola $y = x^2$ meets the cuspidal cubic $y^2 = x^3$ with multiplicity 3 at the origin.
- $\mathbb{Z}_{(p)}$ has no coefficient field (a subfield of $\mathbb{Q}$ has characteristic 0, but $k = \mathbb{F}_p$), nor does $\mathbb{R}[x]_{(x^2+1)}$ ($k = \mathbb{C}$, but -1 is not a square in $\mathbb{R}(x)$); the DVR $\mathbb{C}[x, y]_{(x)}$ has residue field $\mathbb{C}(y)$ and two coefficient fields $\mathbb{C}(y)$ and $\mathbb{C}(x + y)$.

Summary

- A module has finite length iff it is Noetherian and Artinian; its composition series all have the same length $\ell(M)$ (Jordan–Hölder), which is additive in exact sequences, equals $\dim_{R/\mathfrak{m}} M$ when $\mathfrak{m}M = 0$, and equals $\dim_K M$ when R is local with coefficient field K.
- For rings: Artinian $\Leftrightarrow$ Noetherian of dimension 0 $\Leftrightarrow$ finite length $\Leftrightarrow$ a finite product of Artinian local rings, each with a nilpotent maximal ideal.
- A finitely generated module over a Noetherian ring has finite length iff its support (equivalently, its set of associated primes) consists of maximal ideals; over a Noetherian local ring $(R, \mathfrak{m})$ this means $\mathfrak{m}^n M = 0$ for some n, and for $I \neq R$, $\ell(R/I) < \infty$ iff I is $\mathfrak{m}$-primary.
- Next: Krull's principal ideal theorem and height theorem (Jeffries §7.1).