18.705 Commutative Algebra
Lecture 12

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Recap: dimension and integral extensions

- $\dim R$ is the supremum n over chains of primes $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$, and $\operatorname{height}(\mathfrak{p})$ is the supremum over chains ending at $\mathfrak{p}$; thus $\operatorname{height}(\mathfrak{p}) + \dim R/\mathfrak{p} \leq \dim R$.
- A chain of primes is saturated if no prime can be inserted between consecutive terms; chains of maximal (finite) length are saturated.
- For an integral extension $R \subseteq S$: $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective (lying over), $\dim S = \dim R$ (going up), and $\operatorname{height}(\mathfrak{q}) \leq \operatorname{height}(\mathfrak{q} \cap R)$, with equality when R is a normal domain and S is a domain (going down).
- If $R \subseteq S$ is integral and S is a domain, then S is a field iff R is (Lecture 3).
- Module-finite $=$ integral $+$ algebra-finite (Lecture 1), UFDs are normal (Lecture 5).
- Zariski's lemma (Lecture 2): a field that is a finitely generated K-algebra is a finite extension of K.

Making a pure-power leading term

Lemma (Nagata's trick)

Let A be a domain and $0 \neq f \in A[x_1, \ldots, x_n]$ of degree less than N. The A-algebra automorphism φ of $A[x_1, \ldots, x_n]$ with $\varphi(x_i) = x_i + x_n^{N^{n-i}}$ for $i < n$ and $\varphi(x_n) = x_n$ satisfies $\varphi(f) = ax_n^m + (\text{terms of lower degree in } x_n)$ with $0 \neq a \in A$ and $m \in \mathbb{Z}_{\geq 0}$.

Proof: For each monomial $x^e := x_1^{e_1} \cdots x_n^{e_n}$, expanding $\varphi(x^e) = (x_1 + x_n^{N^{n-1}})^{e_1} \cdots (x_{n-1} + x_n^N)^{e_{n-1}} x_n^{e_n}$ gives $x_n^{e_1 N^{n-1} + \cdots + e_{n-1} N + e_n}$ plus terms of lower degree in x_n. If x^e appears in f then $0 \leq e_i < N$, so this exponent is the integer with base-N digits $e_1, \ldots, e_n$; distinct monomials of f thus give distinct exponents. So if $f = \sum_e a_e x^e$ and m is the largest of these exponents, then $\varphi(f) = \sum_e a_e \varphi(x^e)$ has exactly one x_n^m term, with coefficient $a := a_e \neq 0$.

- Example: $f = x_1 x_2$ and $N = 3$ gives $\varphi(f) = (x_1 + x_2^3)x_2 = x_2^4 + x_1 x_2$.
- Use: if $f(r_1, \ldots, r_n) = 0$ in $R \supseteq A$ and $r_i' := r_i - r_n^{N^{n-i}}$, then $\varphi(f)(r_1', \ldots, r_{n-1}', r_n) = 0$ gives a monic equation for r_n over $A_a[r_1', \ldots, r_{n-1}']$.

A linear change of variables over an infinite field

Lemma

Let K be an infinite field and $0 \neq f \in K[x_1, \ldots, x_n]$ of degree m. For some $c \in K^{n-1}$, the linear automorphism ψ with $\psi(x_i) = x_i + c_i x_n$ for $i < n$ and $\psi(x_n) = x_n$ satisfies $\psi(f) = a x_n^m + (\text{terms of lower degree in } x_n)$ with $0 \neq a \in K$ and $m \leq \mathbb{Z}_{\geq 0}$.

Proof: Write $f = f_0 + \cdots + f_m$ with f_j homogeneous of degree j. Each $\psi(f_j)$ is homogeneous of degree j, so only $\psi(f_m)$ can contain x_n^m, with coefficient $f_m(c_1, \ldots, c_{n-1}, 1)$ (set $x_i = 0$ for $i < n$ and $x_n = 1$). Since $f_m \neq 0$ is homogeneous, $f_m(y_1, \ldots, y_{n-1}, 1) \neq 0$ (distinct monomials of degree m dehomogenize to distinct monomials), and a nonzero polynomial over an infinite field has a non-root $c \in K^{n-1}$.

- Example: $\psi(x_1 x_2) = (x_1 + c x_2) x_2 = c x_2^2 + x_1 x_2$ works for any $c \neq 0$.
- K infinite is needed: $f := x_1^q x_2 - x_1 x_2^q$ vanishes on $\mathbb{F}_q^2$, so if $\psi(x_i) = a_i x_1 + b_i x_2$, then $\psi(f) = (a_1 b_2 - a_2 b_1)(x_1^q x_2 - x_1 x_2^q)$ has no pure power monomials.
- Unlike Nagata's trick, ψ preserves degrees, which is what we want for graded rings.

Noether normalization

Theorem (Noether normalization)

Let A be a domain and $R \supseteq A$ a finitely generated A-algebra. There exist $0 \neq a \in A$ and $z_1, \ldots, z_d \in R$ algebraically independent over A such that R_a is module-finite over $A_a[z_1, \ldots, z_d]$; if $A = K$ is a field, then R is module-finite over $K[z_1, \ldots, z_d]$.

Proof: To the blackboard!

We call $K[z_1, \ldots, z_d] \subseteq R$ a Noether normalization of R. If K is infinite, the z_i can be taken to be K-linear combinations of any given generators of R.

- $K[x, y]/(xy)$ is module-finite over $K[x + y]$, as $x^2 = (x + y)x$ and $y^2 = (x + y)y$, but not over $K[x]$, as y is not integral over $K[x]$.
- For $K = \overline{K}$, a Noether normalization of the coordinate ring of a variety $X \subseteq K^n$ gives a map $X \to K^d$ that is surjective with finite fibers (lying over).

On the blackboard: Noether normalization

Theorem. Let A be a domain and $R = A[r_1, \dots, r_n] \supseteq A$ a finitely generated A-algebra. Then there are $0 \neq a \in A$ and $z_1, \dots, z_d \in R$ algebraically independent over A such that R_a is module-finite over $A_a[z_1, \dots, z_d]$. (If $A = K$ is a field then a is a unit and $R_a = R$.)

Proof. Induction on n; for $n = 0$ take $d = 0$ and $a = 1$. If the r_i are algebraically independent over A (i.e. $A[x_1, \dots, x_n] \to R$ is injective) take $z_i := r_i$ and $a = 1$. Otherwise pick $0 \neq f \in A[x_1, \dots, x_n]$ with $f(r_1, \dots, r_n) = 0$ and $N > \deg f$. By Nagata's trick,
$g := f(x_1 + x_n^{N^{n-1}}, \dots, x_{n-1} + x_n^N, x_n) = a x_n^m + (\text{lower terms in } x_n)$ with $0 \neq a \in A$. Put
$r'_i := r_i - r_n^{N^{n-i}}$ for $i < n$, so that $g(r'_1, \dots, r'_{n-1}, r_n) = f(r_1, \dots, r_n) = 0$ and $R = A[r'_1, \dots, r'_{n-1}, r_n]$. Here $m \geq 1$, since $m = 0$ would give $a = g(r'_1, \dots, r'_{n-1}, r_n) = 0$ in $R \supseteq A$. So $a^{-1}g$ is monic of degree m in x_n with coefficients in $R' := A_a[r'_1, \dots, r'_{n-1}]$, and r_n is integral over R'; thus $R_a = R'[r_n]$ is module-finite over R' (Lecture 1).

By induction applied to the A_a-algebra $R' \supseteq A_a$ (A_a is a domain), there exist $0 \neq b \in A_a$, which we may take in A, and $z_1, \dots, z_d \in R'$ algebraically independent over A_a, hence over A, such that R'_b is module-finite over $A_{ab}[z_1, \dots, z_d]$. Multiplying the z_i by a power of a we may assume $z_i \in R$ (this changes neither $A_{ab}[z_1, \dots, z_d]$ nor the algebraic independence). Now $R_{ab} = (R_a)_b$ is module-finite over R'_b, hence over $A_{ab}[z_1, \dots, z_d]$, since module-finiteness is transitive, and we are done. □

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Noether normalization implies the Nullstellensatz

Corollary (Zariski's lemma, again)

If a field L is a finitely generated K-algebra, then $[L : K] < \infty$.

Proof: Let $K[z_1, \ldots, z_d] \subseteq L$ be a Noether normalization. As L is a field, so is $K[z_1, \ldots, z_d]$ (recap), which forces $d = 0$; thus L is module-finite over K.

So Noether normalization gives a second proof of the Nullstellensatz, via Zariski's lemma $\Rightarrow$ weak Nullstellensatz $\Rightarrow$ strong Nullstellensatz (Lectures 2–3).

For any Noether normalization $K[z_1, \ldots, z_d] \subseteq R$ we have $\dim R = \dim K[z_1, \ldots, z_d]$ (going up), and we will now show that $\dim K[z_1, \ldots, z_d] = d$.

The nonnegative integer $d = \dim R$ is a fundamental invariant of R.

The dimension of an affine domain

An affine domain over K is a finitely generated K-algebra that is a domain.

Theorem

Let R be an affine domain over K with Noether normalization $K[z_1, \ldots, z_d] \subseteq R$. Then (i) every chain of primes in R has length at most d, and (ii) every saturated chain of primes $0 = \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_k = \mathfrak{m}$ from 0 to a maximal ideal $\mathfrak{m}$ of R has length $k = d$. In particular, $\dim R = d$.

Proof: To the blackboard!

- $A := K[z_1, \ldots, z_d]$ is a UFD, so every nonzero prime of A contains a nonzero prime (f) with f irreducible, and after a change of variables $f = az_d^m +$ (lower terms in z_d) with $a \in K^\times$ and $m \geq 1$. Then $K[z_1, \ldots, z_{d-1}] \subseteq A/(f)$ is a Noether normalization with $d-1$ variables.
- Going down applies to $A \subseteq R$, as A is normal (Lecture 5) and R is a domain: every prime $\mathfrak{p}$ of A contained in $\mathfrak{q} \cap A$ is the contraction of a prime $\mathfrak{q}' \subseteq \mathfrak{q}$ of R.

On the blackboard: the dimension of an affine domain

Theorem. Let R be an affine domain over K and $A := K[z_1, \dots, z_d] \subseteq R$ a Noether normalization. Then (i) every chain of primes of R has length $\leq d$, and (ii) every saturated chain of primes $0 = \mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_k = \mathfrak{m}$ from 0 to a maximal ideal has length $k = d$.

Proof. Induction on d. If $d = 0$ then R is a domain integral over the field K, hence a field (Lecture 3), so 0 is its only prime and (i), (ii) hold. Let $d \geq 1$ and let $0 = \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_k$ be a chain of primes with $k \geq 1$ (for (i) we may prepend 0; for (ii) $k = 0$ would make R a field, hence A, which is false).

By incomparability $\mathfrak{p}_1 := \mathfrak{q}_1 \cap A \neq 0$, so $\mathfrak{p}_1$ contains an irreducible $f \in A$, and (f) is prime since A is a UFD. By going down there is a prime $\mathfrak{q}' \subseteq \mathfrak{q}_1$ of R with $\mathfrak{q}' \cap A = (f)$, and $\mathfrak{q}' \neq 0$.

Claim: $R/\mathfrak{q}'$ is an affine domain with a Noether normalization in $d - 1$ variables.

Replacing z_i by $z_i - z_d^{N^{d-i}}$ for $i < d$ (which does not change A) with $N > \deg f$ we have $f = a z_d^m + (\text{lower terms in } z_d)$ with $a \in K^\times$, by Nagata's trick; $m \geq 1$ as f is not a unit. Let $A' := K[z_1, \dots, z_{d-1}]$. Then $A/(f)$ is module-finite over A' (it is generated by $1, z_d, \dots, z_d^{m-1}$), and $A' \cap (f) = 0$: a nonzero multiple fh has z_d-degree $m + \deg_{z_d} h \geq 1$, since the leading coefficient a of f is a unit. So $A' \to A/(f) \to R/\mathfrak{q}'$ are both module-finite (note $(f) = \mathfrak{q}' \cap A$), proving the claim.

(i) The chain $\mathfrak{q}'/\mathfrak{q}' \subseteq \mathfrak{q}_1/\mathfrak{q}' \subsetneq \mathfrak{q}_2/\mathfrak{q}' \subsetneq \cdots \subsetneq \mathfrak{q}_k/\mathfrak{q}'$ in $R/\mathfrak{q}'$ has length $\geq k - 1$, so $k - 1 \leq d - 1$ by induction. (ii) If the chain is saturated with $\mathfrak{q}_k = \mathfrak{m}$ maximal, then $\mathfrak{q}' = \mathfrak{q}_1$ (nothing lies strictly between 0 and $\mathfrak{q}_1$), and $0 \subsetneq \mathfrak{q}_2/\mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{m}/\mathfrak{q}_1$ is a saturated chain from 0 to a maximal ideal of $R/\mathfrak{q}_1$, so $k - 1 = d - 1$ by induction. □

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Consequences for affine domains

Recall that the transcendence degree $\operatorname{trdeg}_K L$ of a field extension L/K is the size of any maximal algebraically independent subset of L, unchanged by algebraic extensions.

Corollary

1. $\dim K[x_1, \ldots, x_d] = d$.
2. *If R is an affine domain over K then* $\dim R = \operatorname{trdeg}_K \operatorname{Frac}(R)$.
3. *If R is a finitely generated K-algebra with n generators then* $\dim R \leq n$; *if R is a domain, equality holds if and only if the generators are algebraically independent, equivalently, if and only if* $R \simeq K[x_1, \ldots, x_n]$.

Proof: (1) $K[x_1, \ldots, x_d]$ is its own Noether normalization. (2) If $A \subseteq R$ is module-finite then $\operatorname{Frac}(A) \subseteq \operatorname{Frac}(R)$ is algebraic, so for a Noether normalization $A = K[z_1, \ldots, z_d]$ we get $\operatorname{trdeg}_K \operatorname{Frac}(R) = \operatorname{trdeg}_K K(z_1, \ldots, z_d) = d = \dim R$. (3) $R \simeq K[x_1, \ldots, x_n]/I$, and chains in $V(I)$ have length $\leq n$ by (1); if R is a domain and $I \neq 0$, prepending 0 to a chain in $V(I)$ shows that $\dim R \leq n - 1$.

Heights in affine domains

Corollary

Let R be an affine domain over K of dimension d.

1. *Every maximal ideal of R has height d.*
2. *R is catenary: all saturated chains from $\mathfrak{p}$ to $\mathfrak{q}$ have length $\operatorname{height}(\mathfrak{q}) - \operatorname{height}(\mathfrak{p})$.*
3. *$\operatorname{height}(I) + \dim R/I = d$ for every proper ideal I of R.*

Proof: (1) A chain from 0 to $\mathfrak{m}$ of maximal length is saturated, hence has length d. (2) Fix a saturated chain C from $\mathfrak{q}$ to a maximal ideal. For any saturated chain C' from 0 to $\mathfrak{q}$, $C'C$ is saturated from 0 to a maximal ideal, so $|C'| = d - |C|$; as some C' has length $\operatorname{height}(\mathfrak{q})$, all do. For $\mathfrak{p} \subseteq \mathfrak{q}$, concatenate saturated chains from 0 to $\mathfrak{p}$ and from $\mathfrak{p}$ to $\mathfrak{q}$. (3) For $\mathfrak{p}$ prime, concatenating a saturated chain from 0 to $\mathfrak{p}$ with a longest chain from $\mathfrak{p}$ to a maximal ideal gives $\operatorname{height}(\mathfrak{p}) + \dim R/\mathfrak{p} = d$; in general use $\operatorname{height}(I) = \min_{\mathfrak{p} \in \operatorname{Min}(I)} \operatorname{height}(\mathfrak{p})$ and $\dim R/I = \max_{\mathfrak{p} \in \operatorname{Min}(I)} \dim R/\mathfrak{p}$.

In general only $\operatorname{height}(\mathfrak{p}) + \dim R/\mathfrak{p} \leq \dim R$; (1) and (3) can fail for non-domains.

Examples

- **Hypersurfaces.** Let $f \in A := K[x_1, \ldots, x_n]$ be nonconstant. The minimal primes of (f) are the (g) with g an irreducible factor of f, each of height 1 (a nonzero prime $\mathfrak{p} \subseteq (g)$ contains an irreducible h, and $g \mid h$ forces $(h) = (g)$), so $\operatorname{height}(f) = 1$ and $\dim A/(f) = n - 1$.

- **A non-example.** $R := K[x, y, z]/(xz, yz)$ is the coordinate ring of the plane $z = 0$ together with the z-axis: $\operatorname{Min}(R) = \{(z), (x, y)\}$ with $\dim R/(z) = 2$ and $\dim R/(x, y) = 1$, so $\dim R = 2$, but the maximal ideal $\mathfrak{m} := (x, y, z - 1)$ has height $1 \neq \dim R - \dim R/\mathfrak{m}$. Here $K[x - z, y - z] \subseteq R$ is a Noether normalization (check!) over which the minimal prime (x, y) lies over the nonzero prime $(x - y)$: going down fails.

Generic freeness

For A a Noetherian domain and M a finitely generated A-module, M_a is free over A_a for some $0 \neq a \in A$. Nagata's trick extends this to finitely generated A-algebras.

Theorem (generic freeness)

Let A be a Noetherian domain, R a finitely generated A-algebra, and M a finitely generated R-module (e.g. $M = R$). Then M_a is a free A_a-module for some $0 \neq a \in A$.

Proof: To the blackboard!

Corollary

Let $A \subseteq R$ be domains with A Noetherian and R finitely generated over A. Then the image of $\operatorname{Spec} R \to \operatorname{Spec} A$ contains a nonempty open set $D(a)$.

Proof: Choose a as in the theorem and write $1 = \sum_i c_i e_i$ in a basis (e_i) of R_a over A_a. If the c_i lay in a maximal ideal $\mathfrak{n}$ of A_a then $1 \in \mathfrak{n}R_a$, but $R_a/\mathfrak{n}R_a \simeq \bigoplus_i A_a/\mathfrak{n} \neq 0$; so $\sum_i d_i c_i = 1$ for some d_i, and $e_i \mapsto d_i$ is an A_a-linear retraction $R_a \to A_a$. Thus $\operatorname{Spec} R_a \to \operatorname{Spec} A_a$ is surjective (Lecture 11), and its image in $\operatorname{Spec} A$ is $D(a)$.

On the blackboard: generic freeness

Theorem. Let A be a Noetherian domain, R a finitely generated A-algebra, and M a finitely generated R-module. Then M_a is a free A_a-module for some $0 \neq a \in A$.

Proof. Writing R as a quotient of $B := A[z_1, \dots, z_n]$, M is a finitely generated B-module, so it suffices to prove by induction on n:

$(*)_n$: for every finitely generated module M over $B = A[z_1, \dots, z_n]$ there is $0 \neq a \in A$ with M_a free over A_a.

$(*)_0$ is the Lecture 9 result. Let $n \geq 1$ and take a prime filtration $0 = M_0 \subsetneq \cdots \subsetneq M_k = M$ with $M_i/M_{i-1} \simeq B/P_i$ (Lecture 9). If a_i works for B/P_i then $a := a_1 \cdots a_k$ works for M: each $0 \to (M_{i-1})_a \to (M_i)_a \to (B/P_i)_a \to 0$ is exact with $(B/P_i)_a$ free, hence split, so $M_a \simeq \bigoplus_i (B/P_i)_a$ is free. Thus we may assume $M = B/P$ with P prime. There are three cases:

- $P = 0$: then B is free over A (monomial basis); take $a = 1$.
- $P \cap A \neq 0$: any $0 \neq a \in P \cap A$ gives $(B/P)_a = 0$.
- $P \neq 0$ and $P \cap A = 0$: pick $0 \neq f \in P$ and $N > \deg f$. In the coordinates $z_i' := z_i - z_n^{N^{n-i}}$ ($i < n$), z_n of B, Nagata's trick gives $f = a z_n^m + (\text{lower terms in } z_n)$ with $0 \neq a \in A$, and $m \geq 1$ since $f \notin A$. So f/a is monic in z_n over $B' := A_a[z_1', \dots, z_{n-1}']$, and B_a/fB_a is a free B'-module with basis $1, z_n, \dots, z_n^{m-1}$. As $f \in P$, $(B/P)_a = B_a/PB_a$ is a quotient of B_a/fB_a, hence a finitely generated B'-module, and $(*)_{n-1}$ over the Noetherian domain A_a gives $0 \neq c \in A_a$, which we may take in A, with $(B/P)_{ac}$ free over A_{ac}. □

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Summary

- Noether normalization: every finitely generated K-algebra R is module-finite over a subring $K[z_1, \ldots, z_d]$; for R a domain, $d = \dim R = \operatorname{trdeg}_K \operatorname{Frac}(R)$.
- Geometrically: every affine variety of dimension d admits a finite surjective map to K^d, which can be taken to be a linear projection.
- Affine domains are well behaved: every maximal ideal has height $d = \dim R$, all saturated chains between two primes have the same length, and $\operatorname{height}(I) + \dim R/I = d$. Hypersurfaces in K^n have dimension $n - 1$.
- Generic freeness: a finitely generated module over a finitely generated algebra over a Noetherian domain A becomes free after inverting one element of A; for domains $A \subseteq R$, the image of $\operatorname{Spec} R \to \operatorname{Spec} A$ contains a nonempty open set.
- Next: Artinian rings (Noetherian rings of dimension 0), then Krull's height theorem (Chapter 7).