

18.705 Commutative Algebra

Lecture 11

Adapted from the [2018 lecture notes](#) of Jack Jeffries (§6.1-6.2)

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Krull dimension and height

Definition

A **chain of primes** of length n in R is a sequence $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of primes of R .

- The **(Krull) dimension** $\dim R$ is the supremum of the lengths of chains of primes.
- The **height** $\text{height}(\mathfrak{p})$ of a prime $\mathfrak{p}$ is the supremum of the lengths of chains ending at $\mathfrak{p}$.
- The **height** of an ideal $I \neq R$ is $\text{height}(I) := \inf\{\text{height}(\mathfrak{p}) : \mathfrak{p} \in \text{Min}(I)\}$.

Geometrically, primes correspond to irreducible closed subsets $V(\mathfrak{p})$ of $\text{Spec } R$ (check!), reversing inclusions: $\dim R$ is the supremum of the lengths of chains of irreducible closed subsets, and $\text{height}(\mathfrak{p})$ is the **codimension** of $V(\mathfrak{p})$.

- $\dim K = 0$ for a field K ; in general $\dim R = 0$ iff every prime of R is maximal.
- $\dim \mathbb{Z} = 1$; more generally, a PID that is not a field has dimension 1.
- $\dim K[x_1, \dots, x_d] \geq d$, via $(0) \subsetneq (x_1) \subsetneq \cdots \subsetneq (x_1, \dots, x_d)$; equality holds (later).

Basic properties of dimension and height

- $\text{height}(\mathfrak{p}) = \dim R_{\mathfrak{p}}$, as the primes of $R_{\mathfrak{p}}$ correspond to the primes $\mathfrak{q} \subseteq \mathfrak{p}$; we also have $\dim W^{-1}R \leq \dim R$.
- $\dim R/I$ is the supremum of the lengths of chains of primes in $V(I)$; for $\mathfrak{p}$ prime, $\dim R/\mathfrak{p}$ is that of the chains starting at $\mathfrak{p}_0 = \mathfrak{p}$.
- $\text{height}(\mathfrak{p}) = 0$ iff $\mathfrak{p} \in \text{Min}(R)$, and $\dim R = \sup_{\mathfrak{m} \in \text{Max}(R)} \text{height}(\mathfrak{m}) = \sup_{\mathfrak{p} \in \text{Min}(R)} \dim R/\mathfrak{p}$.
- Concatenating chains gives $\text{height}(\mathfrak{p}) + \dim R/\mathfrak{p} \leq \dim R$, possibly strictly.
- For $\mathfrak{p} \in \text{Min}(R)$ and $f \notin \mathfrak{p}$, prepending $\mathfrak{p}$ to chains in $V(\mathfrak{p} + (f))$ shows $\dim R/(\mathfrak{p} + (f)) < \dim R/\mathfrak{p}$, when finite.
- Noetherian rings can have infinite dimension (Nagata), but their primes have finite height (Krull, later), so Noetherian local rings have finite dimension.

The image of $\operatorname{Spec} S \rightarrow \operatorname{Spec} R$

Let $\varphi: R \rightarrow S$ be a ring map. For ideals $I \subseteq R$ and $J \subseteq S$ we write $IS := \varphi(I)S$ and $J \cap R := \varphi^{-1}(J)$. Recall that the fiber of $\varphi^*: \operatorname{Spec} S \rightarrow \operatorname{Spec} R$ over $\mathfrak{p}$ is Spec of the **fiber ring** $\kappa_\varphi(\mathfrak{p}) := \varphi(R \setminus \mathfrak{p})^{-1}(S/\mathfrak{p}S)$, and that $\kappa(\mathfrak{p}) := \kappa_{\operatorname{id}}(\mathfrak{p}) = \operatorname{Frac}(R/\mathfrak{p})$.

Lemma (image criterion)

A prime $\mathfrak{p}$ of R lies in the image of φ^* if and only if $\mathfrak{p}S \cap R = \mathfrak{p}$.

Proof: If $\mathfrak{q} \cap R = \mathfrak{p}$ then $\mathfrak{p}S \subseteq \mathfrak{q}$, so $\mathfrak{p} \subseteq \mathfrak{p}S \cap R \subseteq \mathfrak{q} \cap R = \mathfrak{p}$. Conversely, if $\mathfrak{p}S \cap R = \mathfrak{p}$ then $R/\mathfrak{p} = R/(\mathfrak{p}S \cap R) \hookrightarrow S/\mathfrak{p}S$, and localizing at $R \setminus \mathfrak{p}$ gives $\kappa(\mathfrak{p}) \hookrightarrow \kappa_\varphi(\mathfrak{p})$ (exactness). So $\kappa_\varphi(\mathfrak{p}) \neq 0$ has a prime: the fiber over $\mathfrak{p}$ is nonempty.

- $\mathfrak{p}S$ need not be prime: for $R = \mathbb{C}[x^n] \subseteq S = \mathbb{C}[x]$ and $\mathfrak{p} = (x^n - 1)R$ we have $\mathfrak{p}S = \prod_{i=0}^{n-1} (x - \zeta_n^i)S$, but each factor contracts to $\mathfrak{p}$.
- If $R \subseteq S$ has an R -linear retraction $\rho: S \rightarrow R$ (R is a **direct summand** of S), then $IS \cap R = I$ for every ideal I of R , so $\operatorname{Spec} S \rightarrow \operatorname{Spec} R$ is surjective.

Extension and contraction along integral maps

Definition

Let I be an ideal of R and S an R -algebra. An element $s \in S$ is **integral over I** if $s^n + a_1 s^{n-1} + \cdots + a_n = 0$ for some $n \geq 1$ and $a_i \in I^i$. The **integral closure $\bar{I}^S$** of I in S is the set of such s , and $\bar{I} := \bar{I}^R$.

We have $I \subseteq \bar{I} \subseteq \sqrt{I}$ (as $s^n \in I$).

Lemma (extension–contraction)

Let $R \subseteq S$ be integral and I an ideal of R . Then $IS \subseteq \bar{I}^S$, hence $IS \cap R \subseteq \bar{I} \subseteq \sqrt{I}$. In particular $\mathfrak{p}S \cap R = \mathfrak{p}$ for every prime $\mathfrak{p}$ of R .

Proof: The determinant trick; to the blackboard!

Example: for $I = (x^2)$ in $R = K[x^2, x^3] \subseteq S = K[x]$ we have $x^3 \in IS \cap R$ but $x^3 \notin I$ (as $x \notin R$); indeed $x^3 \in \bar{I}$, as $(x^3)^2 - (x^2)^3 = 0$ with $(x^2)^3 \in I^2$.

On the blackboard: extension and contraction along integral maps

Lemma. Let $R \subseteq S$ be integral and I an R -ideal. Every $x \in IS$ is integral over I and $IS \cap R \subseteq \sqrt{I}$.

Proof. Write $x = \sum_{k=1}^t \alpha_k s_k$ with $\alpha_k \in I$ and $s_k \in S$. The subring $S' := R[s_1, \dots, s_t]$ is algebra-finite and integral over R , hence module-finite (Lecture 1), and $x \in IS'$; replacing S by S' we may assume $S = Rb_1 + \dots + Rb_u$ is module-finite.

Then $IS = Ib_1 + \dots + Ib_u$, so $xb_i \in IS$ gives $xb_i = \sum_j \alpha_{ij} b_j$ with $\alpha_{ij} \in I$; that is, $(xI_u - A)v = 0$ for $A := (\alpha_{ij})$ and $v := (b_1, \dots, b_u)^T$. Multiplying by the adjugate of $xI_u - A$ gives $\det(xI_u - A)b_j = 0$ for all j (the determinant trick), and since $1 \in \sum_j Rb_j$ we get $\det(xI_u - A) = 0$.

Expanding the determinant, $\det(xI_u - A) = x^u + c_1 x^{u-1} + \dots + c_u$, where c_i is $(-1)^i$ times the sum of the principal $i \times i$ minors of A , each a sum of products of i entries of A ; thus $c_i \in I^i$, and x is integral over I .

Finally, if $x \in IS \cap R$ then $x^u = -(c_1 x^{u-1} + \dots + c_u) \in I$, so $x \in \sqrt{I}$. □

Presented on the blackboard; not part of the projected slides.

Lying over

Theorem (lying over)

If $R \subseteq S$ is an integral extension, then $\operatorname{Spec} S \rightarrow \operatorname{Spec} R$ is surjective: for every prime $\mathfrak{p}$ of R there is a prime $\mathfrak{q}$ of S *lying over* $\mathfrak{p}$, meaning $\mathfrak{q} \cap R = \mathfrak{p}$.

Proof: $\mathfrak{p}S \cap R = \mathfrak{p}$ by the extension–contraction lemma, so $\mathfrak{p}$ is in the image by the image criterion.

- $K[x] \subseteq K[x, y]/(y^2 - x)$ is integral, and the projection of the parabola $y^2 = x$ to the x -axis is surjective.
- $K[x] \subseteq K[x, y]/(xy - 1)$ is not integral, and the projection of the hyperbola $xy = 1$ to the x -axis misses $x = 0$.
- For a number field K , the ring extension $\mathbb{Z} \subseteq \mathcal{O}_K$ is integral, so every prime p has a prime of $\mathcal{O}_K$ above it (which divides the ideal $p\mathcal{O}_K$, as we shall see).

Incomparability

Theorem (incomparability)

Let $\varphi: R \rightarrow S$ be integral, and $\mathfrak{q} \subseteq \mathfrak{q}'$ primes of S with $\mathfrak{q} \cap R = \mathfrak{q}' \cap R$. Then $\mathfrak{q} = \mathfrak{q}'$.

Proof: Let $\mathfrak{p} := \mathfrak{q} \cap R = \mathfrak{q}' \cap R$. Then $R/\mathfrak{p} \hookrightarrow S/\mathfrak{q}$ is an integral extension of domains, and $\mathfrak{q}'/\mathfrak{q}$ is a prime of $S/\mathfrak{q}$ with $(\mathfrak{q}'/\mathfrak{q}) \cap (R/\mathfrak{p}) = 0$. If $\mathfrak{q}' \neq \mathfrak{q}$, a nonzero $s \in \mathfrak{q}'/\mathfrak{q}$ has a nonzero multiple $st \in R/\mathfrak{p}$ (Lecture 3), and $st \in (\mathfrak{q}'/\mathfrak{q}) \cap (R/\mathfrak{p}) = 0$, a contradiction.

Equivalently: every fiber ring $\kappa_\varphi(\mathfrak{p})$ is zero-dimensional. Consider the non-integral extension $R \subseteq R[x]$: the fiber over $\mathfrak{p}$ contains $\mathfrak{p}[x] \subsetneq \mathfrak{p}[x] + (x)$.

Corollary

If $\varphi: R \rightarrow S$ is integral, then $\text{height}(\mathfrak{q}) \leq \text{height}(\mathfrak{q} \cap R)$ for every $\mathfrak{q} \in \text{Spec } S$; in particular, $\dim S \leq \dim R$.

Proof: Each chain $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n = \mathfrak{q}$ contracts to $\mathfrak{q}_0 \cap R \subsetneq \cdots \subsetneq \mathfrak{q}_n \cap R = \mathfrak{q} \cap R$.

Fibers of integral maps are finite

Corollary

Let $\varphi: R \rightarrow S$ be integral.

1. If S is Noetherian, only finitely many primes of S lie over each prime of R .
2. If $S = Rf_1 + \cdots + Rf_n$ then at most n primes of S lie over any prime of R .

Proof: (1) The primes over $\mathfrak{p}$ are the primes of the Noetherian ring $\kappa_\varphi(\mathfrak{p})$ (a localization of a quotient of S), all minimal by incomparability, and Min of a Noetherian ring is finite. (2) $\kappa_\varphi(\mathfrak{p})$ is spanned over $\kappa(\mathfrak{p})$ by the images of the f_i , and an n -dimensional algebra over a field has at most n maximal ideals (check!).

- $\mathbb{Z}[i] = \mathbb{Z} + \mathbb{Z}i$: the fiber over p is $\text{Spec } \mathbb{F}_p[x]/(x^2 + 1)$, which has one point if $p = 2$ or $p \equiv 3 \pmod{4}$ and two points if $p \equiv 1 \pmod{4}$.
- $\mathbb{C}[x, y]/(y^2 - x) = \mathbb{C}[x] + \mathbb{C}[x]y$: the fiber over $(x - a)$ is $\text{Spec } \mathbb{C}[y]/(y^2 - a)$, which has two points if $a \neq 0$ and one (nonreduced) point if $a = 0$.

Going up

Theorem (going up)

Let $\varphi: R \rightarrow S$ be integral, $\mathfrak{p} \subsetneq \mathfrak{p}'$ primes of R , and $\mathfrak{q}$ a prime of S lying over $\mathfrak{p}$. Then some prime $\mathfrak{q}' \supsetneq \mathfrak{q}$ of S lies over $\mathfrak{p}'$.

Proof: $R/\mathfrak{p} \hookrightarrow S/\mathfrak{q}$ is an integral extension (injective since $\mathfrak{q} \cap R = \mathfrak{p}$), so some prime $\mathfrak{q}'/\mathfrak{q}$ of $S/\mathfrak{q}$ lies over $\mathfrak{p}'/\mathfrak{p}$. Then $\mathfrak{q}' \supseteq \mathfrak{q}$ lies over $\mathfrak{p}'$, and $\mathfrak{q}' \neq \mathfrak{q}$ since $\mathfrak{p}' \neq \mathfrak{p}$.

Corollary

Let $R \subseteq S$ be integral. Then $\dim S = \dim R$, and $\dim S/\mathfrak{q} = \dim R/(\mathfrak{q} \cap R)$ for every $\mathfrak{q} \in \operatorname{Spec} S$.

Proof: $\dim S \leq \dim R$ by incomparability. Lying over gives $\mathfrak{q}_0$ over $\mathfrak{p}_0$, and going up (inductively) lifts a chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ to a chain $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n$ with $\mathfrak{q}_i \cap R = \mathfrak{p}_i$; so $\dim S \geq \dim R$. The same argument starting from a given $\mathfrak{q}$, together with incomparability for chains in $V(\mathfrak{q})$, gives $\dim S/\mathfrak{q} = \dim R/(\mathfrak{q} \cap R)$.

Going down

Lemma

Let R be a normal domain with fraction field K , $S \supseteq R$ a domain, and $s \in S$ integral over R , with minimal polynomial $f \in R[t]$. If s is a root of a monic $g \in R[t]$ whose nonleading coefficients lie in a prime $\mathfrak{p}$ of R , all nonleading coefficients of f lie in $\mathfrak{p}$.

Proof: Let $s = s_1, \dots, s_u$ be the roots of f in a splitting field L , and $S' := R[s_1, \dots, s_u] \subseteq L$, which is integral over R . Each s_j is a root of g , as $f \mid g$, so $s_j^{\deg g} \in \mathfrak{p}S'$ and $s_j \in \sqrt{\mathfrak{p}S'}$. The coefficients of f are polynomials in the s_j without constant term, so they lie in $\sqrt{\mathfrak{p}S'} \cap R = \sqrt{\mathfrak{p}S' \cap R} = \sqrt{\mathfrak{p}} = \mathfrak{p}$ (extension-contraction).

Theorem (going down)

Let $R \subseteq S$ be an integral extension of domains with R normal, $\mathfrak{p}' \subsetneq \mathfrak{p}$ primes of R , and $\mathfrak{q}$ a prime of S lying over $\mathfrak{p}$. Then some prime $\mathfrak{q}' \subsetneq \mathfrak{q}$ of S lies over $\mathfrak{p}'$.

Proof: To the blackboard!

On the blackboard: going down

Theorem. Let $R \subseteq S$ be an integral extension of domains with R normal, $p' \subsetneq p$ primes of R , and q a prime of S with $q \cap R = p$. Then some prime $q' \subsetneq q$ of S satisfies $q' \cap R = p'$.

Proof. Let $W := \{rs : r \in R \setminus p', s \in S \setminus q\}$, a multiplicative subset of S containing $R \setminus p'$ and $S \setminus q$.

Claim: $W \cap p'S = \emptyset$. Granting this, the ring $W^{-1}(S/p'S)$ is nonzero, so it has a prime, which corresponds to a prime q' of S with $q' \supseteq p'S$ and $q' \cap W = \emptyset$ (Lecture 4). Then $q' \cap (S \setminus q) = \emptyset$ gives $q' \subseteq q$; $q' \cap (R \setminus p') = \emptyset$ gives $q' \cap R \subseteq p'$; and $q' \supseteq p'S$ gives $q' \cap R \supseteq p'$. So $q' \cap R = p' \neq p = q \cap R$, and $q' \subsetneq q$.

Proof of claim: Suppose $x = rs \in p'S$ with $r \in R \setminus p'$ and $s \in S \setminus q$. Let $g(t) = t^n + v_1 t^{n-1} + \cdots + v_n$ be the minimal polynomial of s over $K := \text{Frac}(R)$; then $g \in R[t]$, since s is integral over R and R is normal (Lecture 5). The polynomial

$$h(t) := r^n g(t/r) = t^n + r v_1 t^{n-1} + \cdots + r^n v_n \in R[t]$$

is monic and irreducible with $h(x) = r^n g(s) = 0$, so it is the minimal polynomial of x . By the extension-contraction lemma $x \in p'S$ is integral over p' , so the lemma gives $r^i v_i \in p'$, and $r \notin p'$ forces $v_i \in p'$. Hence $s^n = -(v_1 s^{n-1} + \cdots + v_n) \in p'S \subseteq pS \subseteq q$, so $s \in q$, a contradiction. $\square$

Presented on the blackboard; not part of the projected slides.

Consequences of going down, and when it fails

Corollary

Let $R \subseteq S$ be an integral extension of domains with R normal. Then $\text{height}(\mathfrak{q}) = \text{height}(\mathfrak{q} \cap R)$ for every $\mathfrak{q} \in \text{Spec } S$.

Proof: $\leq$ is incomparability, and going down lifts a chain ending at $\mathfrak{q} \cap R$ to one ending at $\mathfrak{q}$.

The corollary need not hold when S is not a domain:

- $R = K[x] \subseteq S = K[x, y]/(xy, y^2 - y) \simeq K[x] \times K$ is integral ($y^2 = y$);
- $\mathfrak{q} = (1 - y) = K[x] \times 0$ is a minimal prime lying over (x) , as $x = x(1 - y) \in \mathfrak{q}$;
- nothing lies below $\mathfrak{q}$ over (0) ;
- so $\text{height}(\mathfrak{q}) = 0 < 1 = \text{height}(\mathfrak{q} \cap R)$.

Going down needs normality

- Let $R := \{f \in S : f(0,0) = f(1,0)\} = K[x(1-x), x^2(1-x), y, xy]$, where $S := K[x, y]$; so $\text{Spec } R$ is the plane with $(0,0)$ and $(1,0)$ glued together.
- $R \subseteq S = R[x]$ is integral, and S is the normalization of R : $x \in \text{Frac}(R) \setminus R$ is a root of $t^2 - t + x(1-x)$. The two points of $\text{Spec } S$ over the glued point are $\mathfrak{q}_0 := (x, y)$ and $\mathfrak{q}_1 := (x-1, y)$.
- Let $\mathfrak{p} := \mathfrak{q}_1 \cap R$ and $\mathfrak{p}' := xS \cap R \subsetneq \mathfrak{p}$, the prime of R corresponding to the line $x=0$ (which passes through $(0,0)$ but not $(1,0)$).
- Going down from $\mathfrak{q}_1$ over $\mathfrak{p} \supsetneq \mathfrak{p}'$ fails: a prime $\mathfrak{q}' \subseteq \mathfrak{q}_1$ lying over $\mathfrak{p}'$ contains $x(1-x)$ and xy but not x (as $x \notin \mathfrak{q}_1$), hence contains $x-1$ and y ; so $\mathfrak{q}' = \mathfrak{q}_1$, which lies over $\mathfrak{p} \neq \mathfrak{p}'$.
- Geometrically: the irreducible closed set $V(\mathfrak{p}')$ through the glued point lifts to the line $x=0$, which passes through $\mathfrak{q}_0$ but not $\mathfrak{q}_1$. Normality rules out such gluing.

Summary: what integral maps do to Spec

Let $\varphi: R \rightarrow S$ be integral.

- $\varphi^*: \operatorname{Spec} S \rightarrow \operatorname{Spec} R$ is a closed map: $\varphi^*(V(J)) = V(J \cap R)$, by lying over applied to $R/(J \cap R) \subseteq S/J$. It is surjective if φ is injective.
- Its fibers are zero-dimensional (incomparability), finite if S is Noetherian, and of size at most n if S is generated by n elements as an R -module.
- $\operatorname{height}(\mathfrak{q}) \leq \operatorname{height}(\mathfrak{q} \cap R)$ and $\dim S \leq \dim R$, with $\dim S = \dim R$ if φ is injective (going up): e.g. $\dim \mathcal{O}_K = 1$ and $\dim K[x, y]/(y^2 - x^3) = 1$.
- $\operatorname{height}(\mathfrak{q}) = \operatorname{height}(\mathfrak{q} \cap R)$ if φ is injective, R is a normal domain, and S is a domain (going down).
- For an integral extension of finitely generated K -algebras, maximal ideals lie over maximal ideals, so the induced map of varieties is surjective with finite fibers.