

18.705 Commutative Algebra

Lecture 10

Andrew V. Sutherland

Adapted from the [2018 lecture notes](#) of Jack Jeffries (§5.2-5.3)

October 20, 2026

Recap: minimal and associated primes

Throughout, R is a Noetherian ring. For an R -module M :

- $\text{Ass}(M) = \{\mathfrak{p} = \text{ann}_R(m) : m \in M\} = \{\mathfrak{p} : R/\mathfrak{p} \hookrightarrow M\}$; it is nonempty iff $M \neq 0$, and finite when M is finitely generated; $\bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}$ is the set of zerodivisors on M .
- $\text{Min}(I) \subseteq \text{Ass}(R/I) \subseteq V(I)$, where $\text{Min}(I)$ is the set of minimal primes of I ; the primes in $\text{Ass}(R/I) \setminus \text{Min}(I)$ are **embedded**.
- $\text{Ass}(L) \subseteq \text{Ass}(M) \subseteq \text{Ass}(L) \cup \text{Ass}(N)$ for $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ exact.
- Ass localizes: $\text{Ass}_{W^{-1}R}(W^{-1}M) = \{W^{-1}\mathfrak{p} : \mathfrak{p} \in \text{Ass}(M), \mathfrak{p} \cap W = \emptyset\}$.

The **saturation** $I^{W\text{-sat}} := \{r \in R : wr \in I \text{ for some } w \in W\}$ is the contraction $IW^{-1}R \cap R$ of the extension of I (here I is an ideal, $W \subseteq R$ is a multiplicative set).

Prime avoidance

Lemma (prime avoidance)

*Let $I_1, \dots, I_n, J$ be ideals of R , with all but at most two of the I_i prime.
If $J \subseteq I_1 \cup \dots \cup I_n$, then $J \subseteq I_i$ for some i .*

Proof: To the blackboard!

There is a version for homogeneous ideals of a graded ring; see Jeffries, Lemma 5.23.

Corollary

*Let R be noetherian, M a finitely generated R -module, and I an ideal of R .
If every element of I is a zerodivisor on M , then $Im = 0$ for some $0 \neq m \in M$.*

Proof: $I \subseteq \bigcup_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}$, a finite union of primes, so $I \subseteq \mathfrak{p} = \text{ann}_R(m)$ for some $\mathfrak{p} \in \text{Ass}(M)$.

Equivalently: if $I \not\subseteq \mathfrak{p}$ for every $\mathfrak{p} \in \text{Ass}(M)$, then I contains a nonzerodivisor on M .

On the blackboard: prime avoidance

Lemma. Let $I_1, \dots, I_n, J$ be ideals with I_i prime for $i > 2$. If $J \not\subseteq I_i$ for all i , then $J \not\subseteq \bigcup_i I_i$.

Proof. Induction on n ; for $n = 1$ there is nothing to show. For $n \geq 2$, the induction hypothesis, applied to the $n - 1$ ideals I_j with $j \neq i$ (at most two of which are not prime), gives for each i an element $a_i \in J \setminus \bigcup_{j \neq i} I_j$. If some $a_i \notin I_i$ we are done, so assume $a_i \in I_i$ for every i , and consider

$$a := a_n + a_1 a_2 \cdots a_{n-1} \in J.$$

If $a \in I_i$ for some $i < n$: since $a_i \in I_i$ we get $a_1 \cdots a_{n-1} \in I_i$, hence $a_n \in I_i$, contradicting the choice of a_n . If $a \in I_n$: since $a_n \in I_n$ we get $a_1 \cdots a_{n-1} \in I_n$. For $n = 2$ this says $a_1 \in I_2$, a contradiction; for $n > 2$ the ideal I_n is prime, so $a_i \in I_n$ for some $i < n$, a contradiction. Thus $a \in J \setminus \bigcup_i I_i$. $\square$

Remark. Two non-primes are allowed because the case $n = 2$ does not use primality, but three are not: in $R = \mathbb{F}_2[x, y]/(x, y)^2$ the ideal $\mathfrak{m} = (x, y) = \{0, x, y, x + y\}$ is the union of the three ideals (x) , (y) , $(x + y)$, none of which contains $\mathfrak{m}$. On the other hand, if R contains an infinite field k , no primality is needed at all: a k -vector space is never a finite union of proper subspaces (check!).

Presented on the blackboard; not part of the projected slides.

Primary ideals

Definition

An ideal $\mathfrak{q} \subsetneq R$ is **primary** if $xy \in \mathfrak{q}$ implies $x \in \mathfrak{q}$ or $y \in \sqrt{\mathfrak{q}}$, i.e. if every zerodivisor of $R/\mathfrak{q}$ is nilpotent. If $\mathfrak{q}$ is primary then $\sqrt{\mathfrak{q}}$ is prime: $xy \in \sqrt{\mathfrak{q}} \Rightarrow x^n y^n \in \mathfrak{q} \Rightarrow x^n \in \mathfrak{q}$ or $y^n \in \sqrt{\mathfrak{q}} \Rightarrow x \in \sqrt{\mathfrak{q}}$ or $y \in \sqrt{\mathfrak{q}}$. We say $\mathfrak{q}$ is **p-primary** if $\mathfrak{q}$ is primary and $\sqrt{\mathfrak{q}} = \mathfrak{p}$.

- Primes are primary. If $\sqrt{\mathfrak{q}} = \mathfrak{m}$ is maximal then $\mathfrak{q}$ is $\mathfrak{m}$ -primary ($\mathfrak{m}/\mathfrak{q}$ is the only prime of $R/\mathfrak{q}$, so nonunits of $R/\mathfrak{q}$ are nilpotent); for example, $\mathfrak{m}^n$ is $\mathfrak{m}$ -primary.
- In a UFD, (a) is primary if and only if $a = 0$ or $(a) = (f^n)$ for some prime element f and $n \geq 1$ (check!).
- In $K[x, y]$, $\mathfrak{q} = (x^2, xy)$ is not primary, even though $\sqrt{\mathfrak{q}} = (x)$ is prime: $xy \in \mathfrak{q}$, but $x \notin \mathfrak{q}$ and $y \notin \sqrt{\mathfrak{q}}$.

Characterizations of primary ideals

Proposition

Let R be a Noetherian ring and $\mathfrak{q} \subsetneq R$ an ideal. The following are equivalent:

1. $\mathfrak{q}$ is primary;
2. every zerodivisor of $R/\mathfrak{q}$ is nilpotent;
3. $\text{Ass}(R/\mathfrak{q})$ is a singleton;
4. $\mathfrak{q}$ has exactly one minimal prime and no embedded primes;
5. $\mathfrak{p} := \sqrt{\mathfrak{q}}$ is prime and $\mathfrak{q}$ is saturated with respect to $W := R \setminus \mathfrak{p}$, i.e. $\mathfrak{q} = \mathfrak{q}^{W\text{-sat}}$;
6. $\mathfrak{p} := \sqrt{\mathfrak{q}}$ is prime and $\mathfrak{q}R_{\mathfrak{p}} \cap R = \mathfrak{q}$.

When any of these equivalent conditions hold we have $\text{Ass}(R/\mathfrak{q}) = \{\sqrt{\mathfrak{q}}\}$.

Proof: To the blackboard!

Only $(1) \Leftrightarrow (2) \Leftrightarrow (5) \Leftrightarrow (6)$ holds for arbitrary rings: in the Puiseux series ring $R = \bigcup_{n \geq 1} \mathbb{C}[[x^{1/n}]]$ of Lecture 9 the ideal (x) is primary but $\text{Ass}(R/(x)) = \emptyset$.

On the blackboard: characterizations of primary ideals

Proposition. Let R be Noetherian and $q \subsetneq R$. TFAE: (1) q is primary; (2) every zerodivisor of R/q is nilpotent; (3) $|\text{Ass}(R/q)| = 1$; (4) q has one minimal prime and no embedded primes; (5) $p := \sqrt{q}$ is prime and $q = q^{W\text{-sat}}$ for $W = R \setminus p$; (6) $p := \sqrt{q}$ is prime and $qR_p \cap R = q$.

Proof. (1) $\Leftrightarrow$ (2): y is a zerodivisor on R/q iff $xy \in q$ for some $x \notin q$, and y is nilpotent on R/q iff $y \in \sqrt{q}$. So (2) says precisely that $xy \in q$ and $x \notin q$ imply $y \in \sqrt{q}$.

(2) $\Leftrightarrow$ (3): The zerodivisors of R/q form $\bigcup_{p \in \text{Ass}(R/q)} p$, and the nilpotents form $\bigcap_{p \in \text{Min}(q)} p = \bigcap_{p \in \text{Ass}(R/q)} p$, since $\text{Min}(q) \subseteq \text{Ass}(R/q) \subseteq V(q)$. So (2) holds iff this union equals this intersection, iff all the associated primes coincide; and $\text{Ass}(R/q) \neq \emptyset$ because $R/q \neq 0$.

(3) $\Leftrightarrow$ (4): $\text{Ass}(R/q) \supseteq \text{Min}(q) \neq \emptyset$, and (4) says that $\text{Ass}(R/q) = \text{Min}(q)$ is a singleton.

(1) $\Leftrightarrow$ (5): If q is primary then $\sqrt{q} = p$ is prime, and " $w x \in q$ with $w \notin p$ implies $x \in q$ " is the definition of primary, so $q = q^{W\text{-sat}}$. Conversely, if (5) holds and $xy \in q$ with $y \notin \sqrt{q} = p$, then $y \in W$ and $x \in q^{W\text{-sat}} = q$.

(5) $\Leftrightarrow$ (6): $qR_p \cap R = q^{W\text{-sat}}$ for $W = R \setminus p$ (Lecture 4).

Finally, if (1)–(6) hold then $\text{Ass}(R/q) = \text{Min}(q) = \{\sqrt{q}\}$, as $\sqrt{q}$ is the unique minimal prime of q . $\square$

Presented on the blackboard; not part of the projected slides.

Primary decomposition

Lemma

For ideals $I_1, \dots, I_n$ of R we have $\text{Ass}(R/(I_1 \cap \dots \cap I_n)) \subseteq \bigcup_j \text{Ass}(R/I_j)$.

In particular, a finite intersection of $\mathfrak{p}$ -primary ideals is $\mathfrak{p}$ -primary.

Proof: $R/(I_1 \cap I_2) \hookrightarrow R/I_1 \oplus R/I_2$, so $\text{Ass}(R/(I_1 \cap I_2)) \subseteq \text{Ass}(R/I_1) \cup \text{Ass}(R/I_2)$; induct on n . If the I_j are $\mathfrak{p}$ -primary, $\mathfrak{q} = \bigcap I_j$ has $\sqrt{\mathfrak{q}} = \mathfrak{p}$ and $\emptyset \neq \text{Ass}(R/\mathfrak{q}) \subseteq \{\mathfrak{p}\}$, so $\mathfrak{q}$ is $\mathfrak{p}$ -primary.

Definition

A **primary decomposition** of I is an expression $I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_n$ with each $\mathfrak{q}_i$ primary. It is **minimal** if the $\sqrt{\mathfrak{q}_i}$ are distinct and no $\mathfrak{q}_i$ contains $\bigcap_{j \neq i} \mathfrak{q}_j$.

Merging terms with equal radicals and dropping redundant terms ensures minimality.

Example: in $K[x, y]$, $(x^2, xy) = (x) \cap (x, y)^2 = (x) \cap (x^2, y)$ are two minimal primary decompositions (check!): the (x) -primary component is the same in both, but the (x, y) -primary components differ.

Existence of primary decompositions

Theorem

If R is Noetherian, every ideal of R has a (minimal) primary decomposition.

Proof: To the blackboard!

Geometrically, $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ refines $V(I) = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_n)$, where $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$: besides the irreducible components of $V(I)$ it records **embedded components**, which are invisible in $V(I)$, and the “nilpotent fuzz” $\mathfrak{q}_i \subsetneq \mathfrak{p}_i$ along each component. For example, $V(x^2, xy) \subseteq \mathbb{A}^2$ is the line $x = 0$ with an embedded point at the origin.

Primary decompositions need not exist in non-Noetherian rings. When one does exist, the radicals of the components in a minimal decomposition are uniquely determined.

On the blackboard: existence of primary decompositions

Theorem. If R is Noetherian, every ideal of R has a primary decomposition.

Proof. Call an ideal $q \subsetneq R$ **irreducible** if it is not the intersection of two ideals strictly containing it.

Step 1: every ideal is a finite intersection of irreducible ideals. If not, since R is Noetherian there is an ideal J maximal among the counterexamples. Then $J \neq R$ (the empty intersection) and J is not irreducible, so $J = J_1 \cap J_2$ with $J_1, J_2 \supsetneq J$; by maximality each J_i is a finite intersection of irreducible ideals, hence so is J , a contradiction.

Step 2: every irreducible ideal q is primary. Suppose q is not primary: there are $x, y \in R$ with $xy \in q$, $x \notin q$ and $y \notin \sqrt{q}$. The ascending chain $(q : y) \subseteq (q : y^2) \subseteq \cdots$ stabilizes, say $(q : y^n) = (q : y^{n+1})$. We claim

$$(q + (y^n)) \cap (q + (x)) = q,$$

which shows that q is reducible, since $y^n \notin q$ (as $y \notin \sqrt{q}$) and $x \notin q$. The inclusion $\supseteq$ is clear. If a lies in the left side, write $a = q + by^n$ with $q \in q$, $b \in R$; then $ya \in qy + (xy) \subseteq q$, so $by^{n+1} \in q$, so $b \in (q : y^{n+1}) = (q : y^n)$, so $by^n \in q$ and $a \in q$. □

Remark. Irreducible ideals are primary, but not conversely: in $K[x, y]$ the (x, y) -primary ideal $(x, y)^2$ equals $(x^2, y) \cap (x, y^2)$ (compare monomials).

Presented on the blackboard; not part of the projected slides.

The first uniqueness theorem

Theorem

Let R be Noetherian and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ a minimal primary decomposition. Then

$$\{\sqrt{\mathfrak{q}_1}, \dots, \sqrt{\mathfrak{q}_n}\} = \text{Ass}(R/I).$$

In particular, the radicals of the components and the number of components are the same for every minimal primary decomposition of I .

Proof: To the blackboard!

The components $\mathfrak{q}_i$ with $\sqrt{\mathfrak{q}_i} \in \text{Min}(I)$ are the **minimal** (or **isolated**) components; the others are the **embedded** components. Minimal components are unique (next theorem), but embedded ones need not be, as $(x^2, xy) = (x) \cap (x, y)^2 = (x) \cap (x^2, y)$ shows.

Compare with prime filtrations (Lecture 9): the quotients $R/\mathfrak{p}_i$ of a prime filtration of R/I include every associated prime of R/I , but may include others: $0 \subsetneq 2\mathbb{Z} \subsetneq \mathbb{Z}$ has quotients $\mathbb{Z}/(0)$ and $\mathbb{Z}/(2)$, while $\text{Ass}_{\mathbb{Z}}(\mathbb{Z}) = \{(0)\}$.

On the blackboard: the first uniqueness theorem

Theorem. Let R be Noetherian and $I = q_1 \cap \cdots \cap q_n$ a minimal primary decomposition, with $p_i := \sqrt{q_i}$. Then $\text{Ass}(R/I) = \{p_1, \dots, p_n\}$.

Proof. $\subseteq$: $R/I \rightarrow \bigoplus_i R/q_i$, so $\text{Ass}(R/I) \subseteq \bigcup_i \text{Ass}(R/q_i) = \{p_1, \dots, p_n\}$, since $\text{Ass}(R/q_i) = \{p_i\}$ by characterization (3).

$\supseteq$: Fix i . By minimality we can pick $x \in \bigcap_{j \neq i} q_j$ with $x \notin q_i$. Then $(I : x) = (q_i : x)$, since $rx \in q_j$ for all $r \in R$ and $j \neq i$. We claim $(q_i : x)$ is p_i -primary. It contains q_i and lies in p_i (if $rx \in q_i$ then $r \in \sqrt{q_i} = p_i$, as $x \notin q_i$), so its radical is p_i ; and if $ab \in (q_i : x)$ with $b \notin p_i$, then $(ax)b \in q_i$ forces $ax \in q_i$, so $a \in (q_i : x)$.

Now $R\bar{x} \simeq R/\text{ann}_R(\bar{x}) = R/(I : x) = R/(q_i : x)$ is a submodule of R/I with $\text{Ass}(R/(q_i : x)) = \{p_i\}$, again by (3), so $p_i \in \text{Ass}(R\bar{x}) \subseteq \text{Ass}(R/I)$. □

Remark. The proof shows $p_i = \sqrt{(I : x)}$ for suitable x . Conversely, for any $x \notin I$ we have $\sqrt{(I : x)} = \bigcap_{j: x \notin q_j} p_j$ (check!), and a prime that is a finite intersection of primes is one of them. So the p_i are exactly the primes of the form $\sqrt{(I : x)}$; this argument does not use that R is Noetherian, which is why the radicals are unique whenever a minimal primary decomposition exists.

Presented on the blackboard; not part of the projected slides.

Localizing primary ideals

Lemma

Let R be Noetherian, $\mathfrak{q}$ a $\mathfrak{p}$ -primary ideal, and $\mathfrak{a}$ a prime of R .

1. If $\mathfrak{p} \not\subseteq \mathfrak{a}$, then $\mathfrak{q}R_{\mathfrak{a}} = R_{\mathfrak{a}}$.
2. If $\mathfrak{p} \subseteq \mathfrak{a}$, then $\mathfrak{q}R_{\mathfrak{a}}$ is a $\mathfrak{p}R_{\mathfrak{a}}$ -primary ideal of $R_{\mathfrak{a}}$, and $\mathfrak{q}R_{\mathfrak{a}} \cap R = \mathfrak{q}$.

Proof: (1) Some $x \in \mathfrak{p} \setminus \mathfrak{a}$ has a power in $\mathfrak{q}$, and x is a unit in $R_{\mathfrak{a}}$. (2) Associated primes localize, so $\text{Ass}_{R_{\mathfrak{a}}}(R_{\mathfrak{a}}/\mathfrak{q}R_{\mathfrak{a}}) = \{\mathfrak{p}R_{\mathfrak{a}}\}$, which says that $\mathfrak{q}R_{\mathfrak{a}}$ is primary with radical $\mathfrak{p}R_{\mathfrak{a}}$. Finally, $\mathfrak{q}R_{\mathfrak{a}} \cap R$ is the saturation of $\mathfrak{q}$ with respect to $R \setminus \mathfrak{a} \subseteq R \setminus \mathfrak{p}$, which is $\mathfrak{q}$ since $\mathfrak{q}$ is saturated with respect to $R \setminus \mathfrak{p}$.

Localization at $\mathfrak{a}$ kills exactly the components of a primary decomposition with radicals not contained in $\mathfrak{a}$, and localization commutes with finite intersections of ideals.

The second uniqueness theorem

Theorem

Let R be Noetherian and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ a minimal primary decomposition, with $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$. If $\mathfrak{p}_i$ is a minimal prime of I , then $\mathfrak{q}_i = IR_{\mathfrak{p}_i} \cap R$. In particular, the minimal components are the same in every minimal primary decomposition of I .

Proof: Localization commutes with finite intersections, so $IR_{\mathfrak{p}_i} = \mathfrak{q}_1 R_{\mathfrak{p}_i} \cap \cdots \cap \mathfrak{q}_t R_{\mathfrak{p}_i}$. For $j \neq i$ we have $\mathfrak{p}_j \not\subseteq \mathfrak{p}_i$, since $\mathfrak{p}_j \neq \mathfrak{p}_i$ and $\mathfrak{p}_i$ is minimal in $V(I)$, so $\mathfrak{q}_j R_{\mathfrak{p}_i} = R_{\mathfrak{p}_i}$ by the lemma, and $IR_{\mathfrak{p}_i} = \mathfrak{q}_i R_{\mathfrak{p}_i}$. Contracting to R gives $IR_{\mathfrak{p}_i} \cap R = \mathfrak{q}_i R_{\mathfrak{p}_i} \cap R = \mathfrak{q}_i$, again by the lemma.

Example: $I = (x^2, xy) \subseteq K[x, y]$ has $\text{Min}(I) = \{(x)\}$, and $IR_{(x)} \cap R = (x)$, since y is a unit in $R_{(x)}$.

For a radical ideal I every component is minimal, and the primary decomposition $I = \bigcap_{\mathfrak{p} \in \text{Min}(I)} \mathfrak{p}$ is unique.

The Krull intersection theorem

Theorem

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then $\bigcap_{n \geq 0} \mathfrak{m}^n = 0$.

Proof: To the blackboard!

- Equivalently, the **$\mathfrak{m}$ -adic topology** on R , with the $\mathfrak{m}^n$ a neighborhood basis of 0, is Hausdorff: R embeds in its completion $\hat{R} = \varprojlim_n R/\mathfrak{m}^n$ (e.g. $K[x]_{(x)} \hookrightarrow K[[x]]$).
- The hypotheses matter: in the non-Noetherian local ring $R = \bigcup_n \mathbb{C}[[x^{1/n}]]$ we have $\mathfrak{m} = \mathfrak{m}^2$, as $x^a = (x^{a/2})^2$, so $\bigcap_n \mathfrak{m}^n = \mathfrak{m} \neq 0$.
- More generally (Krull), for any ideal I of a Noetherian ring R and finitely generated module M , $\bigcap_n I^n M = \{m \in M : (1 - a)m = 0 \text{ for some } a \in I\}$; so $\bigcap_n I^n = 0$ when R is a domain and $I \neq R$, or when I lies in every maximal ideal.

On the blackboard: the Krull intersection theorem

Theorem. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then $\bigcap_{n \geq 0} \mathfrak{m}^n = 0$.

Proof. Let $J := \bigcap_n \mathfrak{m}^n$. We show $J \subseteq \mathfrak{m}J$; then $J = \mathfrak{m}J$, and NAK gives $J = 0$, since J is finitely generated.

Let $\mathfrak{m}J = q_1 \cap \cdots \cap q_t$ be a primary decomposition, and put $\mathfrak{p}_i := \sqrt{q_i}$. It suffices to show that $J \subseteq q_i$ for each i .

If $\mathfrak{p}_i \neq \mathfrak{m}$, pick $x \in \mathfrak{m} \setminus \mathfrak{p}_i$. Then $xJ \subseteq \mathfrak{m}J \subseteq q_i$ with $x \notin \sqrt{q_i}$, so $J \subseteq q_i$ because q_i is primary.

If $\mathfrak{p}_i = \mathfrak{m}$, then each of the finitely many generators of $\mathfrak{m}$ has a power in q_i , so $\mathfrak{m}^N \subseteq q_i$ for some N (check!), and $J \subseteq \mathfrak{m}^N \subseteq q_i$. □

Remark. The same argument proves the general form: for an ideal I of a Noetherian ring R and $J := \bigcap_n I^n$, a primary decomposition of IJ gives $J = IJ$, and then $(1 - \alpha)J = 0$ for some $\alpha \in I$ by the determinant trick (Lecture 5). If R is local and $I \subseteq \mathfrak{m}$, then $1 - \alpha$ is a unit and $J = 0$.

Presented on the blackboard; not part of the projected slides.