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Summary-February24

Summing over the group. Let ρ be a representation of G on V. Because G is finite, one can sum over the group. This is a way to produce something that is invariant.

The simplest examples start with a subspace W of V. The sum $U = \sum_g gW$ of the subspaces gW is invariant, and so is the intersection $T = \bigcap_g gW$: For any group element h, $U = hU$ and $T = hT$.

The reason that these subspaces are invariant subspaces is that, as g runs over the group, so does $g' = hg$, though in a different order.

For example, let $G = S_3$, and let $h = y$. As g runs through the group in the order $1, x, x^2, y, xy, x^2y$, $g' = hg$ runs through G in the order $y, x^2y, xy, 1, x^2, x$.

Therefore $\sum_g hg = sum_g g' = \sum_{g'} g' = \sum_g g$, and

$$hU = \sum_g g'W = \sum_g gW = U$$

Similarly,

$$hT = \bigcap_g hgW = \bigcap_g g'W = \bigcap_g gW = T$$

The next example is averaging an element v of the vector space V. The averaging operation is

$$\widetilde{v} = \frac{1}{|G|} \sum_g gv$$

If h is a group element, then $h\widetilde{v} = \frac{1}{|G|} \sum_g hgv$. We put $g' = hg$: $h\widetilde{v} = \frac{1}{|G|} \sum_g g'v$. As g runs over the group, so does g', in a different order. Therefore the sum $\sum_g g'$ is equal to $\sum_{g'} g'$, and $h\widetilde{v} = \frac{1}{|G|} \sum_g gv = \widetilde{v}$.

However, it may very well happen that $\widetilde{v}$ is the zero vector. So this averaging process isn't always interesting.

The factor $\frac{1}{|G|}$ that appears isn't important. It is there so that, if v happens to be inveriant itself, then $\widetilde{v} = v$

Next, let $[\ ,\ ]$ be a positive definite hermitian form on V. The form is called *invariant* if $[v, w] = [gv, gw]$ for all g. If the form is invariant, the operators ρ_g will be *unitary*.

The averaging process can be used to produce an invariant form from an arbitrary form.

We start with an arbitrary positive definite hermitian form $\{\ ,\ \}$ on V. For instance, we could choose a basis for V and carry the standard hermitian form on $\mathbb{C}^n$ over using the basis. We define a new form $[\ ,\ ]$ by

$$[v, w] = \frac{1}{|G|}\{gv, gw\}$$

This form is positive definite and invariant. To prove that it is invariant, we show that $[v, w] = [hv, hw]$ for all h in G:

$$[hv, hw] = \frac{1}{|G|} \sum_g \{ghv, ghw\} = \frac{1}{|G|} \sum_g \{g'v, g'w\}$$

As g runs over the group, so does $g'' = gh$, though in a different order. Therefore

$$[hv, hw] = \frac{1}{|G|} \sum_{g''} \{g''v, g''w\} = [v, w]$$

V. Proof of Maschke's Theorem The theorem asserts that every representation is a direct sum of irreducible representations. To prove it, we start with a representation ρ on a space V. If there is no proper invariant subspace, then ρ is irreducible. If there is a proper invariant subspace W, we look for a complementary subspace W' such that V is the direct sum $W \oplus W'$. If W' exists, we can apply induction on the dimension to conclude that the restrictions of ρ to W and W' are direct sums of irreducible repreentations, and then ρ will be a sum of irreducibles too.

We choose an invariant positive definite form [,] on V, so that $[v, w] = [\rho_g v, \rho_g w]$ for all g in G and all v, w in V. I hope you have earned that this formula shows that ρ_g are *unitary operators*. (See Proposition 8.6.3 of the text. A unitary operator preserves orthogonality. Therefore, if W is invariant, $W = \rho_g W$, and if W' is the orthogonal space $W^\perp$, then $\rho_g W'$ will be the orthogonal space to $\rho_g W = W$. So $W' = \rho_g W'$, i.e., W' is invariant.

Character table for the icosahedral group.

Let G be the icosahedral group of rotational symmetries of a regular icosahedron or dodecahedron. It is isomorphic to the alternating group A_5. The conjugacy classes were described in 18.701, I hope. They can be identified by the angles of rotation. or by the type of permutation of five indices. I've displayed two permutation representations below. The first is the operation of A_5 on five indices. The second is the operation of the icosahedral group on the six pairs of opposite faces of a dodecahedron. For example, a rotation x by $2\pi/5$ fixes the axis of rotation, i.e., one pair of opposite faces. So $\chi_{f.pr}(x) = 1$. Rotation by $2\pi/3$ fixes no pair of opposite faces. Looking at a picture of the dodecahedron, I can't see the face pairs fixed by a rotation by π about an edge, so the number (2) is in parentheses. It can be seen to be the only possible value by orthogonality with the trivial representation.

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\begin{array}{r|ccccc}
 & (1) & (15) & (20) & (12) & (12) \\
 & 0 & \pi & 2\pi, 3 & 2\pi/5 & 4/5\ (angle) \\
 & (.) & (..)(..) & (...) & (.....) & (.....)\ (perm) \\
\hline
\chi_{perm}: & 5 & 1 & 2 & 0 & 0 \\
\chi_{f.pr.}: & 6 & (2) & 0 & 1 & 1
\end{array}
$$
charsA5

Subtracting the trivial character from χ_{perm} and from $\chi_{f.pr.}$ gives two of the irreducible representations. One also has the representation of 3-space by rotations. Its character can be computed easily. Remember that the trace of rotation by θ on 3-space is $1 + 2\cos\theta$, the 1 resulting from the fact that the rotation fixes its axis.

With this information, the character table is computed easily. It is in the text.