We determine some character tables.

I. Let $G$ be the tetrahedral group of symmetries of a regular tetrahedron, which is also the alternating group $A_4$. Its order is 12. The conjugacy classes were probably discussed in 18.701. Unfortunately, we can’t show the process easily, so we display the table. $Initr$ denotes a rotation by angle $2\pi/3$ about a vertex and $y$ denotes rotation by $\pi$ about an edge. There are four conjugacy classes, so four irreducible characters. Let their dimensions be $d_1, \ldots, d_4$. The formula $|G| = d_1^2 + \cdots + d_4^2$ shows that $d_i = 1, 1, 1, 3$. This determines $\chi_i(1)$ for $i = 1, 2, 3, 4$.

The character $\chi$ is the one corresponding to the operation $A_4$ by permutations of four indices. So $\chi(y)$ is the number of indices fixed by $y$, which is a product of two disjoint transpositions. That number is zero. The character $\chi$ is a sum of irreducible characters, including the trivial character $\chi_1$. This determines $\chi_4$.

Finally, let $\rho$ be the one-dimensional representation of $G$ whose character is $\chi_2$. Then $\chi_2(x)$ is the unique eigenvalue of the one-dimensional operator $\rho_x$. Since $x^3 = 1$, it is also true that $(\rho_x)^3 = 1$ and that $\chi_2(x) = 1$. There are three possibilities: $\chi_2(x) = 1, \omega$ or $\omega^2$, with $\omega = e^{2\pi i/3}$. Moreover, $\chi_2(x^2)$ is the unique eigenvalue of $\rho_x^2$, which is the square of $\chi_2(x)$. The three possibilities for $\chi_2$ are the first three rows of the table.

II. Let $G$ be the dihedral group $D_5$ of symmetries of a regular pentagon. Let $x$ denote rotation by $2\pi/5$, and let $y$ be one of the reflection symmetries. The elements of $G$, grouped into conjugacy classes, are $1, \{x, x^4\}, \{x^2, x^3\}, \{y, xy, x^2y, x^3y, x^4y\}$, and the dimensions of the irreducible characters are 1, 1, 2, 2.

The character $\chi$ in the bottom row is the permutation character in which $G$ operates on the vertices of the pentagon. It is the sum $\chi_1 + \chi_3 + \chi_4$.

The character $\chi_3$ is the one that corresponds to the operation of $G$ on the plane, in which $x$ is rotation by $2\pi/5$ and $y$ is a reflection. The value of the character on $x$ is $\alpha = 2 \cos 2\pi/5 = \zeta + \zeta^{-1}$, with $\zeta = e^{2\pi i/5}$, and $\beta = 2 \cos 4\pi/5 = \zeta^2 + \zeta^3$.

The character table is

III. Let $G$ be an arbitrary finite group. The permutation representation in which $G$ operates on itself by left multiplication is called the regular representation, and its character $\chi_{reg}$ is the regular character. Then $\chi_{reg}(g)$
is the number of elements of $G$ fixed by left multiplication by $g$. That number is zero unless $g$ is the identity, in which case it is the order of the group: $\chi_{\text{reg}}(1) = |G|$. If $\chi_i$ is an irreducible character of $G$, 

$$\langle \chi_{\text{reg}}, \chi_i \rangle = \frac{1}{|G|} \sum_g \chi_{\text{reg}}(g) \chi_i(g) = \frac{1}{|G|} \chi_{\text{reg}}(1) \chi_i(1) + 0 + \cdots + 0 = \frac{1}{|G|} |G| \chi_i(1) = \dim \chi_i$$

Let $d_i$ be the dimension of $\chi_i$. The projection formula $\chi_{\text{reg}} = \sum_i \langle \chi_{\text{reg}}, \chi_i \rangle \chi_i$ shows that $\chi_{\text{reg}} = \sum_i d_i \chi_i$. Therefore $|G| = \dim \chi_{\text{reg}} = \sum_i d_i \dim \chi_i = \sum_i d_i^2$. This proves one part of the Main Theorem!
February 24.

Summing over the group. Let $\rho$ be a representation of $G$ on $V$. Because $G$ is finite, one can sum over the group. This is a way to produce something that is invariant.

The simplest examples start with a subspace $W$ of $V$. The sum $U = \sum_g gW$ of the subspaces $gW$ is invariant, and so is the intersection $T = \bigcap_g gW$: For any group element $h$, $U = hU$ and $T = hT$.

The reason that these subspaces are invariant subspaces is that, as $g$ runs over the group, so does $g' = hg$, though in a different order.

For example, let $G = S_3$, and let $h = y$. As $g$ runs through the group in the order $1, x, x^2, y, xy, x^2y, x$, $g' = hg$ runs through $G$ in the order $y, x^2, y, xy, 1, x^2, x$.

Therefore
$$\sum_{g} hg = \sum_{g'} g' = \sum_{g} g,$$
so does $g'$, in a different order. Therefore the sum $\sum_{g'} g'$ is equal to $\sum_{g} g'$, and $h\tilde{v} = \frac{1}{|G|} \sum_{g} gv = \tilde{v}$.

However, it may very well happen that $\tilde{v}$ is the zero vector. So this averaging process isn’t always interesting.

The factor $\frac{1}{|G|}$ that appears isn’t important. It is there so that, if $v$ happens to be invariant itself, then $\tilde{v} = v$.

Next, let $[\ ,\ ]$ be a positive definite hermitian form on $V$. The form is called invariant if $[v, w] = [gv, gw]$ for all $g$. If the form is invariant, the operators $\rho_g$ will be unitary.

The averaging process can be used to produce an invariant form from an arbitrary form.

We start with an arbitrary positive definite hermitian form $\{\ ,\ \}$ on $V$. For instance, we could choose a basis for $V$ and carry the standard hermitian form on $\mathbb{C}^n$ over using the basis. We define a new form $[\ ,\ ]$ by
$$[v, w] = \frac{1}{|G|} \{gv, gw\}$$
This form is positive definite and invariant. To prove that it is invariant, we show that $[v, w] = [hv, hw]$ for all $h$ in $G$:
$$[hv, hw] = \frac{1}{|G|} \sum_g \{ghv, ghw\} = \frac{1}{|G|} \sum_g \{g'v, g'w\}$$
As $g$ runs over the group, so does $g'' = gh$, though in a different order. Therefore
$$[hv, hw] = \frac{1}{|G|} \sum_{g''} \{g''v, g''w\} = [v, w]$$

V. Proof of Maschke’s Theorem The theorem asserts that every representation is a direct sum of irreducible representations. To prove it, we start with a representation $\rho$ on a space $V$. If there is no proper invariant subspace, then $\rho$ is irreducible. If there is a proper invariant subspace $W$, we look for a complementary subspace $W'$ such that $V$ is the direct sum $W \oplus W'$. If $W'$ exists, we can apply induction on the dimension to conclude that the restrictions of $\rho$ to $W$ and $W'$ are direct sums of irreducible representations, and then $\rho$ will be a sum of irreducibles too.

We choose an invariant positive definite form $\{\ ,\ \}$ on $V$, so that $[v, w] = [\rho_g v, \rho_g w]$ for all $g$ in $G$ and all $v, w$ in $V$. I hope you have earned that this formula shows that $\rho_g$ are unitary operators. (See Proposition 8.6.3.
A unitary operator preserves orthogonality. Therefore, if $W$ is invariant, $W = \rho_g W$, and if $W'$ is the orthogonal space $W^\perp$, then $\rho_g W'$ will be the orthogonal space to $\rho_g W = W$. So $W' = \rho_g W'$, i.e., $W'$ is invariant.

**Character table for the icosahedral group.**

Let $G$ be the icosahedral group of rotational symmetries of a regular icosahedron or dodecahedron. It is isomorphic to the alternating group $A_5$. The conjugacy classes were described in 18.701, I hope. They can be identified by the angles of rotation, or by the type of permutation of five indices. I’ve displayed two permutation representations below. The first is the operation of $A_5$ on five indices. The second is the operation of the icosahedral group on the six pairs of opposite faces of a dodecahedron. For example, a rotation $x$ by $2\pi/5$ fixes the axis of rotation, i.e., one pair of opposite faces. So $\chi_{f.pr}(x) = 1$. Rotation by $2\pi/3$ fixes no pair of opposite faces. Looking at a picture of the dodecahedron, I can’t see the face pairs fixed by a rotation by $\pi$ about an edge, so the number $(2)$ is in parentheses. It can be seen to be the only possible value by orthogonality with the trivial representation.

\[
\begin{array}{cccccc}
(1) & (15) & (20) & (12) & (12) \\
0 & \pi & 2\pi,32\pi /5 & 4/5 \ (angle) \\
\end{array}
\]

\[
(0.0.3) \begin{array}{cccccc}
\chi_{perm} : & 5 & 1 & 2 & 0 & 0 \\
\chi_{f.pr.} : & 6 & (2) & 0 & 1 & 1 \\
\end{array}
\]

Subtracting the trivial character from $\chi_{perm}$ and from $\chi_{f.pr.}$ gives two of the irreducible representations. One also has the representation of 3-space by rotations. Its character can be computed easily. Remember that the trace of rotation by $\theta$ on 3-space is $1 + 2 \cos \theta$, the 1 resulting from the fact that the rotation fixes its axis.

With this information, the character table is computed easily. It is in the text.