Summary of February 17 class

GROUP REPRESENTATIONS

G a finite group
V a finite-dimensional (complex) vecctor space

A *representation* of G on V is an operation of G on V by linear operators. So if $g \in G$ and $v \in V$, then gv is another element of V. The properties of the operation are

- $g(hv) = (gh)v$ if g, h are in G
- $1_G v = v$

These are the properties of any operation of a group on a set. In addition, we want the operaton to be linear. (Otherwise we wouldn't be thinking of V as a vector space.) So we require

- $g(v + w) = gv + gw$ if v, w are in V
- $g(cv) = c(gv)$, $c \in \mathbb{C}$.

Let $GL(V)$ denote the group of invertible linear operators on V. Say that V has dimension n. The group $GL(V)$ becomes isomorphic to the general linear group of $n \times n$ matrices when one chooses a basis for V, by

$$\varphi \leftrightarrow \text{matrix of } \varphi$$

Representations are usually denoted by a greek letter, often by ρ (rho). The representation above becomes the map $G \to GL(V)$ defined by: $\rho_g =$ "how g operates", i.e., $\rho_g(v) = gv$. (The element g is in the subscript position to keep it out of the way.)

The linear operator ρ_g is invertible because, as follows from the properties of an operation, $g^{-1}gv = v$. It also follows from those properties that $\rho_g \rho_h = \rho_{gh}$. So ρ is a homomorphism $G \to GL(V)$. Conversely, a homomorphism $G \to GL(V)$ defines a representation.

Example: We use our usual example in which G is the symmetric group S_3, which is isomorphic to the alternating group D_3 of symmetries of a triangle. Let V be the plane $\mathbb{C}^2$. The *standard representation* α of G is the one whose matrices (with repect to the standard basis) are

$$A_x = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \quad A_y = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

So, α_x is rotation by angle $2\pi/3$ and α_y is reflection about the e_1-axis, except that we operate on $\mathbb{C}^2$ rather than on $\mathbb{R}^2$.

I hope that this matrix representation is familiar. However, when one changes to another basis, the matrices change to $B_x = Q^{-1}A_xQ$ and $B_y = Q^{-1}A_yQ$. One might not recognize the representation when given by B.

Then how can one classify the representations? The secret method is to look at the *trace* of the operators and forget everything else. The trace of an operator is independent of the basis.

We define the *character* χ (greek letter "chi") of a representation ρ to be

$$\chi(g) = trace \rho_g$$

We write the character of the standard representation, and also of two other representations:

• the it trivial representation τ: In the trivial representation, V is a one-dimensional vector space, and $\tau_g = 1$ (the identity operator) for all g. Its character is $\chi_\tau(g) = 1$ for all g.

• the *sign representation* σ: Here V is also a one-dimensional vector space, and σ_g is the sign of the permutation g in S_3. Its character is $\chi_\sigma(x) = 1$ and $\chi_\sigma(y) = -1$.

We assemble the three characters into a table, listing the values of the character below the elements of G:

	1	x	x^2	y	xy	x^2y
standard :	2	−1	−1	0	0	0
trivial :	1	1	1	1	1	1
sign :	1	1	1	−1	−1	−1

This table has the following general properties:

1. The first column is the dimension of the vector space, which is also called the dimension of the representation and the dimension of the character. By definition,

$$dim\, \rho = dim\, \chi = dim V$$

This is clear. The trace of the identity operator ρ_1 is the dimension of the space V.

2. The characters are constant on the conjugacy classes.
The conjugacy classes are: $\{1\}$, $\{x, x^2\}$, $\{y, xy, x^2y\}$

This is also clear. If g, g' are conjugate group elements, then ρ_g and $\rho_{g'}$ are also conjugate. Conjugate operators (or matrices) have the same trace.

The next properties aren't so clear:

3. The rows are orthogonal.

4. The lengths of the rows is 6, which is also the order of the group.
These properties illustrate the Main Theorem, which is stated below.

For the theorem, we need to know about invariant subspaces and irreducible representations. This is quite simple.

Let ρ be a representation of G on a vector space V. An *invariant subspace* W of V is a subspace such that, if w is an element of W, then $gw = \rho_g w$ is also in W for all g. Or, one might write $gW \subset W$ (in which case $gW = W$ because ρ_g is invertible.)

The representation ρ is *irreducible* if V has no proper invariant subspace. The representations listed above are irreducible. If ρ is an irreducible representation, its character χ is called an *irreducible character*.

We define a pairing on characters, by normalizing the standard hermitian form:

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_g \overline{\chi(g)} \chi'(g)$$

The term $\frac{1}{|G|}$ takes care of the fact that the lengths of the rows in the diagram above is $6 = |G|$.

Main Theorem. • The irreducible characters form an orthonormal basis for the space of *class functions* (functions on the conjugacy classes).
• The number of irreducible characters is equal to the number of conjugacy classes.
• Let $\chi_1, ..., \chi_k$ be the irreducible characters, and let d_i be the dimension of χ_i (the dimension of the vector space V_i). Then $|G| = d_1^2 + \cdots + d_k^2$.