18.702 Comments on Problem Set 7

1. Chapter 14, Problem 1.4. (Schur’s Lemma)
This should have been relatively easy.

2. Chapter 14, Problem 2.4. (ideals that are free modules)
This should have been relatively easy.

3. Chapter 14, Problem 4.5. (lattices in the plane)
Let $L$ denote the subgroup of $\mathbb{C}$ generated by $\alpha, \beta, \gamma$. If the three numbers lie on a line through 0, they don’t span a lattice. Suppose that they don’t lie on a line. Then $L$ will be a lattice if and only if there is a relation $a\alpha + b\beta + c\gamma = 0$ with $a, b, c \in \mathbb{Z}$. The proof is as follows:

Let’s say that $\alpha$ and $\beta$ are independent. Then $\gamma$ will be a real combination $r\alpha + s\beta$ for some real numbers $r, s$. If $r$ and $s$ are rational numbers, and if $d$ is a common denominator for $r$ and $s$, $L$ will be a subgroup of the lattice with basis $\alpha/d, \beta/d$. Then $L$ will be a discrete subgroup of $\mathbb{R}^2$ that contains a basis, and therefore $L$ will be a lattice.

Suppose that $r$ and $s$ aren’t both rational. Let $M$ be the lattice spanned by $\alpha$ and $\beta$. Then no integer multiple of $\gamma$ is in $M$. So if $m$ and $n$ are distinct integers, the cosets $m\gamma + M$ and $n\gamma + M$ are distinct, because $(m - n)\gamma$ is not in $M$. Each coset $m\gamma + M$ contains an element in the parallelogram with vertices 0, $\alpha, \beta, \alpha + \beta$. This gives us infinitely many distinct elements of $L$ in that parallelogram. So $L$ is not discrete, and therefore is not a lattice.

4. Chapter 14, Problem 4.6 (index of a homomorphism)
Changing bases in domain and range, we can change $A$ to a diagonal matrix $A' = Q^{-1}AP$, where $P$ and $A$ are invertible integer matrices. They will have determinants $\pm 1$, so $|\det A|$ will be unchanged. So we can assume that $A$ is diagonal, etc...
Chapter 14, Problem M.5. (matrices that send a lattice to itself)

The matrices that stabilize the standard lattice \( L_0 = \mathbb{Z}^2 \) are the invertible integer matrices, the integer matrices with determinant \( \pm 1 \), the elements of \( GL_2(\mathbb{Z}) \).

If \( L \) is any lattice in \( \mathbb{R}^2 \), there will be an invertible real matrix \( P \) such that \( PL = L_0 \). Then if \( A \) is an invertible real matrix \( A \) such that \( AL = L \), then \( PAP^{-1}L_0 = L_0 \). Therefore \( PAP^{-1} \) is an invertible integer matrix.

This is one answer to the question: The matrices \( A \) that stabilize a lattice are those that are conjugate to an invertible integer matrix.

Next, when is a real matrix \( A \) conjugate to an invertible integer matrix? Since the characteristic polynomials of \( A \) and \( PAP^{-1} \) are equal, the characteristic polynomial of such a matrix will have integer coefficients. It will have the form \( p(t) = t^2 - at \pm 1 \), where \( a = \text{trace}(A) \) is an integer.

We’ll show that conversely, a real matrix \( A \) whose trace is an integer and whose determinant is \( \pm 1 \) is conjugate to an invertible integer matrix. Let \( T \) be the linear operator of multiplication by \( A \) on \( \mathbb{R}^2 \).

We choose a vector \( v_1 \) in \( \mathbb{R}^2 \) that isn’t an eigenvector of \( T \). (This is a trick that was used in 18.701 in the proof that \( PSL_2 \) is simple.) Then \( v_1 \) and \( v_2 = Tv_1 \) are independent, so \( (v_1, v_2) \) is a basis of \( \mathbb{R}^2 \). We write \( Tv_2 \) in terms of this basis: \( Tv_2 = rv_1 + sv_2 \). The matrix of \( T \) with respect to the basis \( (v_1, v_2) \) is then

\[
B = \begin{pmatrix}
0 & r \\
1 & s
\end{pmatrix}
\]

Its characteristic polynomial \( t^2 - st - r \) is also the characteristic polynomial of \( A \). So \( s = \text{trace}(A) \) and \( r = \pm 1 \). Then \( B \) is an invertible integer matrix that is conjugate to \( A \).

The trick assumes that there is a vector \( v_1 \) that isn’t an eigenvector of \( A \). If every vector is an eigenvector, then \( A = cI \). That case is OK too.