18.702 Comments on Problem Set 4

1. Chapter 11, Exc. 7.4. (fractions of formal power series)

The main point is to show that if the constant coefficient $a_0$ of a series $f(x) = a_0 + a_1x + \cdots$ isn't zero, the series is invertible, a unit in the power series ring $F[[x]]$. This is proved by induction, and there are various ways to set the induction up. Since we can multiply by $a_0^{-1}$, we can assume that the constant coefficient of $f$ is 1. An informal approach is to write the inverse with undetermined coefficients and solve the resulting equations:

$$(1 + a_1x + a_2a^2 + \cdots)(1 + b_1x + b_2a^2 + \cdots) = 1$$

The equations are

$$a_1 + b_1 = 0,$$

$$a_2 + a_1b_1 + b_2 = 0,$$

$$a_3 + a_2b_1 + a_1b_2 + b_3 = 0,$$ etc.

The important point is that the coefficient of $x^n$ has the form $E_n + b_n$, where $E_n$ is an expression involving some $a_i$ and some $b_j$ with $j < n$. So one can solve recursively.

3. Chapter 11, Exc. 8.4 (maximal ideals of $\mathbb{R}[x]$)

This is rather simple. The maximal ideals of $\mathbb{R}[x]$ are principal ideals, generated by monic irreducible polynomials. The irreducible polynomials are linear or quadratic. A linear polynomial has a root on the real line. An irreducible quadratic polynomial $q$ has a pair of complex conjugate roots $\alpha, \overline{\alpha}$, one of which is in the upper half plane. We associate that point of the upper half plane to the maximal ideal generated by $q$.

3. Chapter 11, Exc. 9.12 (polynomials without common zeros)

I assigned this so that you would learn that the Nullstellensatz is useful. To write 1 as a combination of $f_1, f_2, f_3$, one can use repeated division with remainder, as in the Euclidean algorithm.

For example, since $f_1$ is monic in $t$, one can use it to divide $f_3$. The remainder is

$$g = f_3 - tf_1 = 4tx^2 + 2t + 1.$$ Then one can divide $g$ by $f_2$, obtaining remainder $h = g - xf_2 = 2t + 4x + 1$. We replace $f_3$ by $\frac{1}{2}h$, which is linear and monic in $t$. Then one can use $h$ to divide $f_1$ and $f_2$, etc.

However, substituting back at the end is a real pain. Sorry.
4. Chapter 12, Exc. 2.8 (division with remainder in $\mathbb{Z}[i]$)

One can divide in $\mathbb{C}$ and take a nearby Gauss integer. For example,

$$
\frac{4 + 36i}{5 + i} = \frac{(4 + 36i)(5 - i)}{26} = \frac{56 + 176i}{26} = \left(2 + \frac{4}{26}\right) + \left(7 - \frac{6}{26}\right)i
$$

So $4 + 36i = (2 + 7i)(5 + i) + r$, where the remainder $r = 4 + 36i - (2 + 7i)(5 + i) = 1 + 4i$. 