Comments on 18.702 Problem Set 3

1. A finite group $G$ operates on itself by conjugation. This operation produces a permutation representation of $G$.

(a) Determine the character $\chi_c$ of this representation.

$\chi_c(g)$ is the number of group elements such that $hgh^{-1} = g$, i.e., such that $hg = gh$. It is the order of the centralizer of $g$: $\chi_c(g) = |Z(g)|$.

(b) Let the conjugacy classes in $G$ be $C_1, \ldots, C_k$, and let $\chi'$ be another character. Write $\langle \chi_c, \chi' \rangle$ as a sum over the conjugacy classes.

Let $g_i$ be an element of $C_i$. Let $c_i = |C_i| = |C(g_i)|$ and $z_i = |Z(g_i)|$. Then $c_iz_i = |G|$. So

$$\langle \chi_c, \chi' \rangle = \frac{1}{|G|} \sum_i c_i \chi_c(g_i) \chi'(g_i) = \frac{1}{|G|} \sum_i c_iz_i \chi'(g_i) = \sum_i \chi'(g_i)$$

(c) Explain how to determine the decomposition of $\chi_c$ into irreducible characters by looking at the character table.

Let $\chi_1, \ldots, \chi_k$ be the irreducible characters, as listed in the character table. Then $\chi_c = r_1\chi_1 + \cdots + r_k\chi_k$, where $r_k$ is the sum of the entries in the row $i$ of the character table.

For example, let $G$ be the tetrahedral group $T$. Looking at the character table 10.4.14 in the text, one sees that $\chi_c = 4\chi_1 + \chi_2 + \chi_3 + 2\chi_4$.

As a check: The dimensions of the characters are 1, 1, 1 and 3, respectively. So $4 \cdot 1 + 1 \cdot 1 + 1\cdot 1 + 2 \cdot 3 = 12$, which is the order of the group.

2. Chapter 10, Exc. M.4 (elements in the center)

Suppose that $z$ is in the center of $G$. Let $\rho$ be a representation of $G$ on $V$, let $a$ be an eigenvalue of $\rho_z$, and let $W$ be the subspace of $V$ of vectors such that $\rho_zw = aw$. This is not the zero space, and it is an invariant subspace. To check this, we must show that for all $g$ in $G$, and all $w$ in $W$, $\rho_gw$ is in $W$, i.e., $\rho_z(\rho_gw) = a(\rho_gw)$. Since $z$ is in the center and $\rho$ is a homomorphism,

$$\rho_z(\rho_gw) = \rho_{zg}w = \rho_{gz}w = \rho_g\rho_zw = \rho_gaw = a\rho_gw$$

So if $\rho$ is an irreducible representation, then $W = V$. This means that $\rho_z$ is multiplication by $a$.

Conversely, let $\rho : G \longrightarrow GL(V)$ be a representation. If $\rho_z$ is multiplication by a scalar, then it is in the center of $GL(V)$. The intersection of the kernels of the irreducible representations is the trivial subgroup $\{1\}$ of $G$. This is true because the kernel of the regular representation...
representation $\rho^{reg}$ is \{1\}, and $\rho^{reg}$ is the direct sum of irreducible representations. Therefore, if $\rho z$ is in the center of $GL(V)$ for every irreducible representation $\rho$, then $z$ is in the center of $G$.

3. Chapter 11, Exc. 3.3 (kernels of some homomorphisms)

(b) Since every ideal of $\mathbb{R}[x]$ is principal, there is a single generator. It must be a polynomial in the kernel that has the lowest possible degree. The polynomial $f(x) = (x - (2 + i))(x - (2 - i)) = x^2 - 4x + 5$ is in the kernel. Since $2 + i$ isn’t real, the kernel can’t contain a linear or a constant polynomial. So $f$ generates the kernel.

(e) The polynomials $u = y - x^2$ and $v = z - x^3$ generate the kernel. To see this, let $g(x, y, z)$ be in the kernel. Since $u$ is monic in $y$, we may do division with remainder in $\mathbb{C}[x, y, z]$: $g = uq + r$, where $r$ has degree zero in $y$, i.e., is independent of $y$, and is in the kernel. Next, we divide $r$ by $v$ in the ring $\mathbb{C}[x, z]$: $r = vq' + r'$, where $r'$ is independent of $y$ and of $z$. It is a polynomial in $x$ in the kernel. Since $x$ maps to $t$, the only such polynomial is the zero polynomial: $r' = 0$. Therefore $g = uq + vq'$.

4. Chapter 11, Exc. 3.9 (unipotent and nilpotent elements)

This is based on the power series expansion $(1 + x)^{-1} = 1 + x + x^2 + \cdots$. If $x^n = 0$, then $(1 + x)^{-1} = 1 + x + x^2 + \cdots + x^{n-1}$.

Let $p$ be a prime. The binomial coefficients $\binom{p}{i}$ for $i = 1, \ldots, p - 1$ are all divisible by $p$. Therefore, in characteristic $p$, $(1 + a)^p = 1 + a^p$, $(1 + a)^{p^2} = (1 + a^p)^p = 1 + a^{p^2}$, etc.