1. Chapter 10, Exc. M.5 (diagonalizing commuting matrices)

Say that $A$ has order $m$ and $B$ has order $n$, and let $G$ be the product $C_m \times C_n$ of cyclic groups, generated by $x$ of order $m$ and $y$ of order $n$.Sending $x^i y^j$ to $A^i B^j$ defines a representation of $G$. Since $G$ is abelian, this representation is a sum of one-dimensional representations. Changing basis to exhibit this decomposition diagonalizes $A$ and $B$.

2. Chapter 11, Exc. 3.4 (a homomorphism $\mathbb{C}[x, y] \to \mathbb{C}[t]$)

Experimenting, one finds that
\[
g(x, y) = y - x^3 + 3x^2 - 3x + 1
\]
is in the kernel. Let $g(x, y)$ be any polynomial in the kernel $K$. Since $f$ is monic in the variable $y$, we can do division with remainder: $g = fq + r$, where $r$ has degree zero in $y$. It is a polynomial in $x$ alone. Since $g$ and $f$ are in $K$, so is $r$. The image of $r(x)$ is $r(t + 1)$, which isn't zero unless $r = 0$. Therefore $r = 0$, and $g$ is in the principal ideal generated by $f$. So $K = (f)$.

Let $I$ be any ideal of $\mathbb{C}[x, y]$. Its image $\overline{I}$ is an ideal in the principal ideal domain $\mathbb{C}[t]$. Let $\overline{p}(t)$ be a generator for $\overline{I}$, and let $p(t)$ be an element that maps to $\overline{p}$. Let $J$ be the ideal of $\mathbb{C}[x, y]$ generated by $p$ and $f$. Then $J$ contains $K$, and its image $\overline{J}$ is equal to $\overline{I}$. By the Correspondence Theorem, $J = I$.

3. Chapter 11, Exc. 3.9 (unipotent and nilpotent elements)

This relies on the power series expansion $(1 + x)^{-1} = 1 + x + x^2 + \cdots$. If $x^n = 0$, then $(1 + x)^{-1} = 1 + x + x^2 + \cdots + x^{n-1}$.

Let $p$ be a prime. The binomial coefficients $\binom{p}{i}$ for $i = 1, \ldots, p - 1$ are all divisible by $p$. Therefore, in characteristic $p$, $(1 + a)^p = 1 + a^p$, $(1 + a)^{p^2} = (1 + a^p)^p = 1 + a^{p^2}$, etc.

4. Chapter 11, Exc. 6.8 (Chinese Remainder Theorem)

Suppose that $I, J$ are comaximal, say that $1 = u + v$ with $u \in I$ and $v \in J$. Then if $x \in I \cap J$, we will have $u = ux + vx$, and both $ux$ and $vx$ will be in $IJ$. So $I \cap J \subset IJ$. The other inclusion $IJ \subset I \cap J$ is true for any ideals.

Now let $a$ and $b$ be given. Then $a = ua + va$, and $va \equiv a$ modulo $I$ and $va \equiv 0$ modulo $J$. Similarly, $ub \equiv b$ modulo $J$ and $ub \equiv 0$ modulo $I$. So $x = va + ub$ works.
5. Chapter 11, Exc. M.3 (maximal ideals in a ring of sequences)

We know some surjective homomorphisms to the field $\mathbb{R}$. Namely, we can send a sequence $a$ to $a_k$ for any $k$, and also to the limit $a_\infty$. Let’s call the kernels of these homomorphisms $M_k$ and $M_\infty$. We show that these are the only maximal ideals.

Let $M$ be any maximal ideal. If $M$ contains any one of those ideals $M_k$, it will be equal to $M_k$ because $M_k$ is maximal. Suppose that $M$ isn’t equal to $M_k$ for some finite $k$. Then for any $k$, $M$ contains elements $a$ with $a_k \neq 0$. Using these elements, one can show that any sequence with limit zero is in $M$:

Let $e$ be the sequence with $e_i = 0$ for $i \neq k$ and $e_k = 1$. Let $c$ denote the constant sequence $a_k^{-1}, a_k^{-1}, \ldots$. The product $cae$ is equal to $e$ and it is in $M$ because $M$ is an ideal. So any finite combination of such sequences $e$ is in $M$.

Since the sequences with limit zero form the ideal $M_\infty$, $M \supset M_\infty$, and since $M_\infty$ is maximal, these two ideals are equal.