1. Chapter 10, Exercise 6.5. (the standard representation of $S_n$)

As with any permutation representation, the trivial representation is a summand, corresponding to the invariant subspace $U$ spanned by $(1,...,1)^t$. Since permutation matrices are orthogonal (and therefore unitary), the space $W$ orthogonal to $U$ is an invariant subspace. It is the space of vectors $(a_1,...,a_n)^t$ such that $\sum a_i = 0$, and its dimension is $n - 1$.

Let $w$ be an nonzero vector in $W$. In 18.701, in problem M1 of Chapter 4, you proved that the span of the orbit of $w$ can have dimension $0, 1, n - 1, or n$, and that the case 1 occurs when the vector is constant: $(a, a, ..., a)^t$. Since $w$ is in $W$ and nonzero, it isn’t a constant vector. So the dimension of the span must be $n - 1$. Every nonzero vector $w$ in $W$ spans $W$. Therefore $W$ is irreducible.

5. Determine the character table of the symmetric group $S_5$.

The table consists of the first seven rows below. The classes to the left of the second vertical line are the even conjugacy classes.

<table>
<thead>
<tr>
<th>size</th>
<th>(1)</th>
<th>(15)</th>
<th>(20)</th>
<th>(24)</th>
<th>(10)</th>
<th>(20)</th>
<th>(30)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(.)</td>
<td>(..)(..)</td>
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<td>(.....)</td>
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<td>(..)(..)</td>
<td>(....)</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>5</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>6</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In the table, $\chi_2$ is the sign character, $\chi_3$ is obtained from the standard permutation representation, which is labelled $\text{perm}$, and $\chi_4 = \chi_2 \chi_3$.

Because there are so many cross-checks in a character table, there is more than one way to determine it.

I. One way is to compute the permutation representation for the operation by conjugation on the conjugacy class (..). Its character is the last row of the table. Using the projection formula, one sees that this character is the sum of three irreducible characters, including $\chi_1$ and $\chi_3$. So (..) = $\chi_1 + \chi_3$ is an irreducible character, the one labelled $\chi_5$, and $\chi_6 = \chi_5 \chi_2$. Then $\chi_7$ is determined by orthogonality.
II. Another method: Let’s split the hermitian product up by writing

$$\langle \chi, \chi' \rangle = \langle \chi, \chi' \rangle^+ + \langle \chi, \chi' \rangle^-,$$

where

$$\langle \chi, \chi' \rangle^+ = \frac{1}{120} \sum_{g \text{ even}} \chi(g) \chi'(g) \quad \text{and} \quad \langle \chi, \chi' \rangle^- = \frac{1}{120} \sum_{g \text{ odd}} \chi(g) \chi'(g)$$

Also, let’s denote the hermitian form on characters $\xi$ and $\xi'$ of $A_5$ by

$$[\xi, \xi'] = \frac{1}{60} \sum_{\text{even}} \xi(g) \xi'(g)$$

Then if $\chi^e$ denotes the restriction to $A_5$ of a character $\chi$ on $S_5$, we will have

$$[\chi^e, \chi'^e] = 2 \langle \chi, \chi' \rangle^+.$$  

The coefficient 2 results from the fact that the factors $\frac{1}{|G|}$ are different for the two groups.

Let $\chi$ be an irreducible character of $S_5$. Then

$$[\chi^e, \chi^e] = 2 \langle \chi, \chi \rangle^+ \leq 2 \langle \chi, \chi \rangle = 2$$

Since $[\chi^e, \chi^e]$ is an integer, it must be 1 or 2.

If $[\chi^e, \chi^e] = 1$, we can make two conclusions: $\chi^e$ is an irreducible character of $A_5$, and $\chi(g) = 0$ for every odd permutation $g$.

If $[\chi^e, \chi^e] = 2$, $\chi^e$ will be a sum of two distinct irreducible characters of $A_5$, and $\chi(g)$ will be nonzero for some odd permutations $g$.

etc..

Notice that the character values are integers. It isn’t hard to verify this directly.

For example, a 4-cycle $g$ is conjugate to its inverse, and therefore $i$ and $-i$ appear as eigenvalues of $\rho_g$ the same number of times. One could make use of this fact when determining the table.