18.702 Comments on Problem Set 1

1. Chapter 10, Exercise 2.3 (irreducible representations of $S_3$)

(a) Let $v = u + xu + x^2u$. Then $xv = v$, so the orbit of $v$ consists of the elements $v$ and $yv$, which may be equal or not. The span of the set $(v, yv)$ will be an invariant subspace $Z$ of dimension at most 2. Then $Z$ is the required space unless $v = 0$, in which case $Z$ is the zero space.

   Next, suppose that $v = 0$. The orbit of $w = u + yu$ consists of the elements $w, xw$ and $x^2w$, and since $v = 0$, the sum $w + xw + x^2w$ is zero. So this orbit spans an invariant subspace of dimension at most 2, and we are done unless $v = 0$ and $w = 0$. If $v = w = 0$, then $yu = -u, xyu = -xu, x^2yu = -x^2u$. So the orbit of $u$ spans the same space as does $(u, xu, x^2u)$. And since $u + xu + x^2u = v = 0$, the span of the orbit of $u$ has dimension at most 2, and it isn’t zero. So in any case, there is a nonzero invariant subspace of dimension at most 2.

2. Chapter 10, Exercise 4.2 (a group of order 55)

The orders of the elements of $G$ can be 1, 5, or 11. The Sylow Theorems tell us that there is a normal subgroup $H$ of order 11. Moreover, all Sylow 11 subgroups are conjugate. So $H$ is the only one. All elements of order 11 are contained in $H$.

   The centralizer of an element $x$ of order 11 contains $H$ and since $G$ isn’t abelian, it isn’t the whole group. So $Z(x) = < x >$, and $|C(x)| = 5$.

   Any element $y$ of $G$ that isn’t contained in $H$ must have order 5, and we must have $Z(y) = < y >$ and $|C(y)| = 11$.

   The class equation is $55 = 1 + 5 + 5 + 11 + 11 + 11 + 11$.

   The quotient group $G/H$ is cyclic of order 5. It has five irreducible representations of dimension 1, and combining them with the canonical map $G \to G/H$ gives us five representations of $G$ of dimension 1.

   Thus there are seven irreducible representations, of dimensions 1, 1, 1, 1, 1, 5, 5.
3. Chapter 10, Exercise 4.3c (character table for $D_4$)

3’. Chapter 10, Exercise 4.10 (completing a character table)

(a) There is one missing character, of dimension 4:

$$\chi_5 \quad 4 \quad -1 \quad 0 \quad 0 \quad 0$$

It is determined by orthogonality. An easy way to determine this character is to use the fact that the columns of the table are orthogonal.

(b) The order of the conjugacy class of an element $g$ determines the order of the centralizer by the counting formula. Since any element $g$ centralizes $g$, the order of $g$ divides the order of the centralizer. Thus the order of $a$ is 5, and the orders of $b, c, d$ are either 2 or 4. Since $\chi_4(b) = i$, the order of $b$ is 4. Similarly, $c$ has order 4. Then $b^2$ has order 2, so there must be an element of order 2. Therefore the order of $d$ is 2.

(c,d,e) The kernel of the representation $\rho_2$ is a normal subgroup $H$ of order 10. Since $C_{10}$ has only two elements of order 2 while the kernel has 5 such elements, $H = D_5$. If $N$ is any proper normal subgroup, we can take a nontrivial representation of the quotient group: $G/N \rightarrow GL(V)$. The composition of maps $G \rightarrow G/N \rightarrow GL(V)$ will be a representation of $G$, and $N$ will be in its kernel. So $N$ must be a subgroup of the kernel of one of the representations of $G$. The only other normal subgroup is the kernel of $\rho_3$, which is a cyclic group of order 5.
4. Chapter 10, Exercise M.1 (vibrations of a molecule)

Let 
\[ R_x = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \quad \text{and} \quad R_y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

be the matrices that represent the standard representation of the dihedral group \( G \). Then in block form,

\[ S_x = \begin{pmatrix} 0 & 0 & R_x \\ R_x & 0 & 0 \\ 0 & R_x & 0 \end{pmatrix} \quad \quad S_y = \begin{pmatrix} R_y & 0 & 0 \\ 0 & 0 & R_y \\ 0 & R_y & 0 \end{pmatrix} \]

The space \( V \) decomposes into irreducible representations as follows:

\( W_1 : (v_1, v_2, v_3) = (v, v, v), \) \( v \) is arbitrary.

This space represents parallel translations of the molecule.

\( W_2 : (v_1, v_2, v_3) = c(e_2, R_x e_2, R_x^2 e_2), \)

This space represents rotations.

\( W_3 : (v_1, v_2, v_3) = c(e_1, R_x e_1, R_x^2 e_1) \)

This space represents vibrations in which the molecule remains equilateral, but stretches and/or shrinks.

\( W_4 \): The orthogonal space to \( W_1 \oplus W_2 \oplus W_3 \).

This space contains the vector \( w = c(2, 0, -1, -\sqrt{3}, -1, \sqrt{3}) \). It is the span of \( w \) and \( S_x w \).

Chemists would call these four spaces the \textit{modes} of vibration, though they might eliminate the first two, as not being true vibrations.

The fourth mode is confusing. Taking \( c > 0 \), the vector \( w \) represents a vibration in which the atom \( a_1 \) moves horizontally to the right, and \( a_2, a_3 \) compensate by moving in to the left while coming together. At time \( t_0 + \Delta t \), the molecule becomes isosceles.