Comments on Quiz 2

The problems are of equal value.

You are expected to justify your assertions, but you may state and use without proof results from lectures or from the assigned reading, unless you are asked to prove them here.

1. A linear operator $T$ on a complex vector space $V$ has eigenvectors $v_1, v_2, v_3$, with eigenvalues $-1, 0, 1$, respectively. Prove that the subspace spanned by the three vectors has dimension 3.

One needs to prove that the vectors are independent. Suppose $c_1v_1 + c_2v_2 + c_3v_3 = 0$. Then $T(\sum c_i v_i) = -c_1v_1 + c_3v_3 = 0$, and $T^2(\sum c_i v_i) = c_1v_1 + c_3v_3 = 0$. Therefore $c_1v_1 = c_3v_3 = 0$. Since an eigenvector can’t be zero, $c_1 = c_2 = 0$. Then $c_2 = 0$ too.

2. Let $a$ be the vector $(1, 2)^t$, and let $r$ denote reflection about the $e_1$-axis. The isometry $m = t_ar$ is a glide reflection. Determine the glide line of $m$.

The isometry operates as

$$ m \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x+1 \\ -y+2 \end{pmatrix} $$

This operation sends the line $y = 1$ to itself,

$$ m \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x+1 \\ 1 \end{pmatrix} $$

So that is the glide line.

3. Let $G$ be the group of invertible upper triangular $2 \times 2$ real matrices,

$$ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} $$

$a, d \neq 0$, and let $S = \mathbb{R}^2$ be the set of two-dimensional column vectors. The group $G$ operates by multiplication on the set $S$. Decompose $S$ into orbits for this operation.

There are three orbits: $S_1 = \{(0, 0)^t\}$, $S_2 = \{x, 0)^t$ with $x \neq 0\}$, and $S_3 = \{(x, y)^t$ with $y \neq 0\}$.

4. Let $G$ denote the group $GL_2(\mathbb{F}_3)$ of invertible $2 \times 2$ matrices with entries modulo 3. The order of $G$ is 48. Determine the order of the conjugacy class of the element $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $G$.

The centralizer of $A$ consists of the invertible matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. There are 4 such matrices, so the conjugacy class has order 12. Too many of you were careless when counting the elements of the centralizer.