18.701 Comments on Quiz 1

The problems are of equal value.

Please show your work and justify your assertions.

1. Decide whether the permutation \((1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)\) is odd or even.

   It is an even permutation.

   A \(k\)-cycle is even if it contains an odd number of indices, and odd if an even number of indices. The sign of the permutation is the product \((-1)(1)(-1) = 1\).

2. Let \(G\) be a cyclic group of order 12. How many of the elements of \(G\) are generators for the group?

   The answer is 4.

   A power \(x^k\) will generate \(G\) if its order in the group is 12, which happens if and only if \(k\) is relatively prime to 12. So \(k\) can be 1, 5, 7, 11.

3. How many elements of order 2 does the symmetric group \(S_4\) contain?

   The answer is 9.

   The elements of order 2 are the transpositions (2-cycles) and the products of disjoint transpositions such as \((1\ 2)(3\ 4)\). There are 6 transpositions and 3 products of transpositions.

4. Let \(G\) be a group. Under what circumstances is the map \(\varphi : G \rightarrow G\) defined by \(\varphi(g) = g^2\) a homomorphism?

   \(\varphi\) will be a homomorphism if and only if \(G\) is abelian.

5. Determine the number of subgroups of index 8 in the symmetric group \(S_4\).

   The number is 4.

   The counting formula is \(|G| = [G : H]|H|\), where \([G : H]\) is the index. Since \(|G| = 4 = 24, |H|\) will have order 3 if the index is 8. A group of order 3 is cyclic, generated by an element of order 3, which will be a 3-cycle. There are 8 three-cycles, and a subgroup of order 3 contains two of them contains.