1. Let $G$ be the group $GL_3(F_2)$. Its Class Equation $168 = 1 + 56 + 24 + 24 + 42 + 21$ was computed in the previous assignment.

(i) Determine the numbers of $p$-Sylow subgroups with $p = 2, 3, 7$.

(ii) Determine the orders of the elements and number of elements of each order.

Conjugate elements have the same order. Since there are six conjugacy classes, the elements of $G$ can have at most six orders, including order 1 for the identity.

The Third Sylow Theorem tells us that the number of Sylow 7-groups is either 1 or 8. Every element of order 7 lies in a Sylow 7-group, distinct Sylow 7-groups intersect only in the identity, and each one contains 6 elements of order 7. So if there were one Sylow 7-group, $G$ would contain 6 such elements. This isn’t possible because no conjugacy class of order $< 6$ other than the class of the identity. So there are 8 Sylow 7-groups, and $G$ contains 48 elements of order 7. The only possibility is that they form the two conjugacy classes of order 24.

There is another way to see this: Let $x$ be an element in one of the classes of order 24. The counting formula tells us that $|G| = |C(x)||Z(x)|$, i.e., $168 = 24|Z(x)|$. Therefore $|Z(x)| = 7$. Since $x$ is an element of $Z(x)$, its order is 7. Therefore the elements in the classes of order 24 have order 7.

Analogous reasoning shows that there are 28 Sylow 3-groups and that the elements of order 3 make up the class of order 56. This leaves two classes, of orders 42 and 21.

The elements of a Sylow 2-group can have orders 1, 2, 4, or 8. If there were an element $x$ of order 8, there would also be elements ($x^2$ and $x^4$) of orders 4 and 2. Then the elements of orders 2, 4, 8 would form at least 3 conjugacy classes. This isn’t possible, so the elements have orders 4 or 2. Since $G$ contains $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ of order 4, those orders do occur.

The elements of order 4 come in pairs $x, x^{-1}$, so there is an even number of them. (In a finite group, the number of elements of order $n$ with $n > 2$ is even.) Therefore there are 42 elements of order 4, and 21 elements of order 2.

Finally, the Third Sylow Theorem tells us that the number of Sylow 2-groups can be 1, 3, 7, or 21. If there were 7 or fewer groups, there wouldn’t be enough elements to fill out the two conjugacy classes. Therefore $G$ contains 21 Sylow 2-groups.

2. Chapter 7, Exercise 8.6. (groups of order 55)

This is analogous to the example of groups of order 21 that is on page 205 of the text.
3. Use the Todd-Coxeter Algorithm to determine the order of the group generated by two elements $x, y$,

(a) with relations $x^3 = 1$, $y^2 = 1$, and $xyx = yxy$.

This is the trivial group.

(b) with relations $x^3 = 1$, $y^3 = 1$, and $xyx = yxy$.

This is a group of order is 24.

4. Chapter 7, Exercise M.1 (groups generated by two elements of order two)

Let $x, y$ be elements of order 2 that generate $G$, and let $z = xy$. Then $x = zy^{-1} = zy$, so $y$ and $z$ also generate $G$, and with generators $y, z$, the relations that are given become $y^2 = 1$ and $zyzy = 1$ or $yz = z^{-1}y$. Using these relations, we can write any word in $y, z, z^{-1}$ as $w = z^i y^j$, where $i$ can be any integer, positive or negative, and $j = 0$ or 1.

The elements of the form $w = z^k y$ have order at most 2 because $w^2 = z^k y z^k y = z^k z^{-k} y y = 1$. Moreover, $z$ and $w$ generate $G$, and $wz = z^{-1} w$. If $w = 1$, then $z = z^{-1}$. In this case, $G$ is generated by a single element $z$ of order 2, and is a cyclic group of order 2.

Let’s suppose that there is no relation $w = 1$, but that $z^k y$ for some integer $k$, and let $n$ be the smallest positive integer such that $z^n = 1$. The relations $z^n = 1, y^2 = 1, yz = z^{-1}y$ define the dihedral group $D_n$. In this case, $G = D_n$.

Summing up: $G$ will be one of these groups: the infinite dihedral group generated by two elements $z, y$ with the relations $y^2 = 1, zyzy = 1$, or a dihedral group $D_n$, or a cyclic group $C_2$. 