1. Chapter 6, Exercise 5.8. \textit{(frieze patterns.)}

There are seven different groups $G$.

An element of $G$ will send the ribbon to itself. It can be a horizontal translation $t_v$, a glide $t_v r$ with horizontal glide vector, the reflection $r$ about the $x$-axis, a rotation with angle $\pi$ about some point on the $x$-axis, or a reflection with some vertical axis of reflection. When coordinates are chosen, a rotation with angle $\pi$ will have the form $t_v \rho$ with $\rho = \rho_\pi$, and a reflection with vertical axis will have the form $t_v s$, where $s$ is reflection about the $y$-axis.

Since it is periodic and discrete, the translation group $L$ of $G$ will have the form $a \mathbb{Z}$, where $a = (a_1, 0)$ is the shortest vector in $L$. Then translations in $G$ will be $\{t_{na}\}$ with $n$ in $\mathbb{Z}$.

Let’s begin by determining the possible point groups of $G$. The homomorphism $M \rightarrow O_2$ drops the translation. It’s kernel is the group of translations $t_v$ with $v \in L$. So the elements of the point group are among the elements $\{1, r, \rho, s\}$ (with bars over the letters to indicate that these elements aren’t considered as elements of $G$). There are five possibilities for the point group:

$G = \{1\}, \{1, r\}, \{1, \rho\}, \{1, s\}, \{1, r, \rho, s\}$.

The rest of the problem consists in analyzing each possibility. For example, suppose that $G = \{1, \rho\}$. Then $\rho$ will be represented by an element $x = t_v r$ in $G$, and we can multiply $x$ on the left by any translation in $G$. Doing so, we can move $v$ into the range $0 \leq v < a$. Then $x^2 = t_v rt_v r = t_v t_v r r = t_{2v}$ is an element of $G$. (Here $rv = v$ because $v$ is horizontal.) Since $x^2$ is in $G$, $2v = na$ for some $n \in \mathbb{Z}$. Since $v$ is in the interval $[0, a)$, there are only two possibilities: $v = 0$ or $v = \frac{1}{2} a$.

The formula $|G| = |\ker||\text{image}|$ shows that $T$ has index 2 in $G$ and that $G = T \cup xT$. So the elements of $G$ are $t_{na}$ and $t_{na} + va$. There are two possibilities in this case.

If $G = \{1, \pi\}$, we may choose coordinates so that $s$ is in $G$. Then $G = T \cup sT$. There is just one possibility in this case.
2. Chapter 6, Exercise 11.1. (operations of $S_3$ on a set of 4.

$S_3 = \{1, x, x^2, y, xy, x^2y\}$, where $x = (1 \ 2 \ 3)$ and $y = (1 \ 2)$. The relations are $x^3 = 1$, $y^2 = 1$ and $y = x^2y$. Let’s denote $S_3$ by $G$.

The way to do this is to consider the ways that the set of four elements decomposes into orbits. There are five possibilities.

1. $4 = 4$: one orbit of order 4. Since the order of an orbit divides the order of $G$, this isn’t possible.

2. $4 = 1 + 1 + 1 + 1$: four orbits of order one. This is the trivial action of $G$.

3. $4 = 1 + 1 + 2$: The group $G$ operates trivially on the orbits of order 1. We must decide whether it can operate nontrivially on a set $\{a, b\}$ of two elements, and if so, in how many ways. Since $x$ has order 3, we can’t have $xa = b$ and $xb = a$, because this would imply $a = 1a = x^3a = x^2(xa) = x^2b = x(xb = xa = b$. So $x$ must fix both $a$ and $b$. Then if the operation is nontrivial, $y$ must operate as the transposition $(a \ b)$. One checks that this is possible by checking that the relations are satisfied. So, up to relabeling the elements of $S$, there is one operation with this orbit decomposition.

4. $4 = 2 + 2$: The operation on each orbit will be as described in case 3. There is one such operation.

5. $1 + 3$: Here $x$ must operate as a 3-cycle on the orbit of three, say as the permutation $(a \ b \ c)$. (If $x$ operates trivially, then there cannot be an orbit of size 3.) Since $yx = x^2y$, $y$ cannot operate trivially. So $y$ is a transposition. Relabeling if necessary, we may suppose that $y = (a \ b)$. There is one such operation.