18.701 Comments on Problem Set 3

1. Chapter 2, Exercise M.6a,b (paths in $\mathbb{R}^k$)

(a) We’ll check transitivity. Let $a, b, c$ be points of $S$, and suppose that there is a path $X(t)$ in $S$ from $a$ to $b$ and a path $Y(t)$ from $b$ to $c$. We must show that there is a path in $S$, say $Z(t)$ that connects $a$ to $c$. The idea is to travel with twice the velocity from $a$ to $b$ and from $b$ to $c$. So the path $Z$ is defined by $Z(t) = X(2t)$ for $0 \leq t \leq \frac{1}{2}$, and $Z(t) = Y(2t - 1)$ for $\frac{1}{2} \leq t \leq 1$. Then $Z(0) = X(0) = a$ and $Z(1) = Y(1) = c$. The path lies entirely in $S$ because $X(t)$ and $Y(t)$ take values in $S$. It is continuous at all points except possibly $t = \frac{1}{2}$, because $X$ and $Y$ are continuous. And at $t = \frac{1}{2}$, it is continuous from the left because $X$ is continuous from the left at $t = 1$, and continuous from the right for the analogous reason. (I don’t care much about precision on this point, but continuity should be mentioned.)

2. (a) If $X(t)$ is a path from $A$ to $B$ in $GL_n$ and $Y(t)$ is a path from $C$ to $D$, then the matrix product $X(t)Y(t)$ defines a path from $AB$ to $CD$. It is continuous because matrix multiplication is continuous.

3. (a) Chapter 2, Exercise M.8 ($SL_n$ is connected)

We know from a previous assignment that $SL_n$ is generated by elementary matrices of the first type: $E = I + aev_{ij}$. They are connected to the identity by a path $E_t = I + ate_{ij}$ in $SL_n$. Then $A$ connects to $EA$ by the path $E_tA$. Since the $\approx$ is an equivalence relation, any two elements of $SL_n$ can be connected by a path.

4. Chapter 3, Exercise 4.4 (order of $GL_2(\mathbb{F}_p)$)

A $2 \times 2$ matrix $A$ is invertible if and only if its columns are independent. To determine two independent vectors $v_1, v_2$, one may choose for $v_1$ any nonzero vector. This gives us $p^2 - 1$ choices for $v_1$. Then once $v_1$ is chosen, we can choose for $v_2$ any vector that is not a multiple of $v_1$. This gives us $p^2 - p$ choices for $v_2$, given $v_1$. So there are $(p^2 - 1)(p^2 - p)$ invertible matrices.

5. Chapter 6, Exercise 11.6 (a homomorphism from $GL_2(\mathbb{F}_3)$ to $S_4$)

I like to denote the elements of $\mathbb{F}_3$ as $0, 1, -1$. Every nonzero vector $X$ in $F^2$ spans a one-dimensional subspace that consists of the three vectors $0, X, -X$, and $-X$ spans the same subspace. Therefore the one-dimensional subspaces are spanned by the four vectors $X_1 = (1, 0)^t, X_2 = (0, 1)^t, X_3 = (1, 1)^t, X_4 = (-1, 1)^t$.

Let’s denote these subspace spanned by $X_i$ by $x_i$. An element $A \in GL_2$ sends $AX_i$ to $\pm X_j$ for some $j$. The permutation $\alpha$ associated to a matrix $A$ in $GL_2$ is determined by the relation that if $AX_i = \pm X_j$, then $\alpha(i) = j$. This is a permutation because, if $i \neq i'$ then, since $A$ is invertible, we can’t have $AX_i = \pm AX_{i'}$.

Next, a product $AB$ operates as “first do $B$, then $A$”, and a product of permutations $\alpha \beta$ operates as “first do $\alpha$, then $\beta$”. So products are sent to products. The map $A \mapsto \alpha$ is a group homomorphism.

If $A$ is sent the identity permutation, then $AX_i = \pm X_i$ for all $i$. This implies that $A = \pm I$. For example, the relation $AX_1 = \pm X_1$ means that the first column of $A$ is $\pm (1, 0)^t$. So the kernel of the homomorphism $\varphi$ has order 2. Since $GL_2$ has order 48, the counting formula tells us that the image of $\varphi$ has order 24. This is also the order of $S_4$. So the map is surjective.