1. Chapter 1, Exercise 6.2: An integer matrix $A$ is invertible and its inverse has integer entries if and only if $\det A = 1$.

The proofs of the two directions are different.

The determinant of an integer matrix is an integer. If $A$ has an integer inverse, the formula $(\det A)(\det A^{-1}) = \det(AA^{-1}) = \det I = 1$ shows that $\det A$ divides 1, and therefore is equal to $\pm 1$.

To show that an integer matrix with determinant 1 has an integer inverse, the simplest thing is to use the formula for the inverse in terms of the cofactor matrix. Another way would be to reduce $A$ to the identity using invertible integer row operations.

2. Chapter 1, Exercise M.8. (an exercise in logic)

(b) There is nothing wrong with the sequence of three steps. If $X$ is a solution of the equation $AX = B$, then $LAX = LB$, and so $X = LB$. However, if $AX \neq B$ there is no reason to expect $X = LB$ to be true. So if the equation has no solution, the sequence of steps can’t be applied. It tells us nothing. In mathematical parlance, the sequence of steps proves uniqueness of the solution: If there is a solution $X$, then $X$ is equal to $LB$.

This shows that we should check our work, because the steps we use may fail to be invertible. And of course, we might have made a mistake.

If $A$ has a right inverse $R$, a matrix such that $AR = I$, then $ARB = B$, so $X = RB$ solves the equation. There may also be other solutions. In mathematical parlance, this is referred to as existence of a solution. Whether or not of a left inverse exists is irrelevant for this conclusion.

One thing that makes this problem confusing is that the lines $AX = B$ and $X = LB$ have different interpretations. When we write $AX = B$, we mean “solve this equation for $X$”, while $X = LB$ is a statement about $X$.

3. Chapter 1, Exercise M.11. (the discrete dirichlet problem)

(c) I assign this problem to teach you about square systems. The system $LX = B$ is square. Theorem 1.2.21 asserts that it has a unique solution for all $B$ if and only if the only solution of the homogeneous equation $LX = 0$ is the trivial solution $X = 0$.

If $X$ solves the homogeneous equation, it is a harmonic function that is equal to zero on the boundary. Then $-X$ is also a harmonic function equal to zero on the boundary. The maximum principle tells us that both $X$ and $-X$ are bounded above by 0, so $X = 0$.
4. Chapter 2, Exercise 4.8b. \((\text{generating } \text{SL}_n(\mathbb{R}))\)

(b) Let’s do the \(2 \times 2\) case. Let \(A\) be a matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) with determinant equal to 1. We must show that \(A\) can be reduced to the identity using the first type of elementary row operations. In describing this process, we’ll use the symbols \(a, b, c, d\) to denote the matrix entries in the matrices we get as we go along. Our end result should be \(a = d = 1, \ b = c = 0\).

If \(c = 0\), then \(a\) can’t be zero. In that case, we add \(\text{row } 1\) to \(\text{row } 2\) to eliminate this possibility. Next, since \(c \neq 0\) in our new matrix, we can add a multiple of \(\text{row } 2\) to \(\text{row } 1\) to change \(a\) to 1. Then we add a multiple of \(\text{row } 1\) to \(\text{row } 2\) to change \(c\) to 0. The new matrix has \(a = 1\) and \(c = 0\). Elementary operations of the first type don’t change the determinant. So the determinant of the new matrix with \(a = 1\) and \(c = 0\) is still equal to 1. Therefore \(d = 1\) in this matrix, and one further row operation reduces the matrix to the identity.

6. \((\text{optional})\) Chapter 2, Exercise M.16. \((\text{the homophonic group})\)

The group is said to be trivial, but I haven’t seen a convincing proof that \(v = 1\). One proof that I don’t want to accept is that, in some dictionaries, the word “civies”, which means civilian clothing when worn by people in the military, can also be spelled with two v’s, as “civvies”.