Plane Crystallographic Groups with Point Group $D_2$

The possibilities for a discrete group $G$ of isometries of the plane whose translation group $L$ is a lattice and whose point group $\overline{G}$ is the dihedral group $D_2$ are described here.

For reference:
- When coordinates are chosen, every isometry can be written as $m = t_v \varphi$, where $\varphi$ is an orthogonal linear operator and $t_v$ is a translation.
- The homomorphism $M \xrightarrow{\pi} O_2$ sends $t_v \varphi$ to $\varphi$. Its kernel is the subgroup of translations in $M$.
- The point group $G$ is the image of $G$ in $O_2$, so $\pi$ defines a surjective homomorphism $G \rightarrow G$ whose kernel is the group of translations in $G$.
- The translation group $L$ is the additive group of vectors $v$ such that $t_v \varphi$ is in $G$. The translation group $L$ is a lattice if it contains two independent vectors.
- The elements of $G$ carry $L$ to $L$.

With suitable coordinates, $G = \{1, r, s, \rho\}$, where $r$ denotes reflection about the horizontal axis, $s$ denotes reflection about the vertical axis, and $\rho$ denotes rotation through the angle $\pi$ about the origin.

The bars over the letters are there to distinguish elements of $G$ from those of $G$. They have no other meaning.

1. Description of the lattice $L$.

Let $u$ be a point of $L$ that is in the first quadrant. Since $G$ operates on $L$, $L$ contains the horizontal vector $u + ru$ as well as the vertical vector $u + su$. So $L$ contains nonzero horizontal and vertical vectors. We choose a horizontal vector $a = (a_1, 0)^t$ in $L$ of minimal positive length. This can be done because $L$ is a discrete subgroup of $\mathbb{R}^2$. Then the horizontal vectors in $L$ are the integer multiples of $a$. Similarly, we choose a vertical vector $b = (0, b_2)^t$ in $L$ of minimal positive length. The vertical vectors in $L$ are the integer multiples of $b$.

Let $L_1$ denote the lattice $a\mathbb{Z} + b\mathbb{Z}$, and let $L_2 = a\mathbb{Z} + c\mathbb{Z}$, where $c = \frac{1}{2}(a + b)$.

**Lemma 1.** Any vector $v$ in $\mathbb{R}^2$, that isn’t in $L_1$ can be written uniquely in the form $v = w + u$, where $w$ is in $L_1$ and $u$ is in the rectangle whose vertices are 0, $a$, $a + b$, and not on the ‘far edges’ $[a, a + b]$, or $[b, a + b]$. If $v$ is in $L$, then $u$ is in the interior of the rectangle.

**proof.** Since $a, b$ are independent, they form a basis of $\mathbb{R}^2$. So $v = xa + yb$ for some $x, y$ in $\mathbb{R}$. We can write $x = m + p$ with $m \in \mathbb{Z}$ and $0 \leq p < 1$, and $y = n + q$ with $n \in \mathbb{Z}$ and $0 \leq q < 1$. Then $w = ma + nb$ is in $L_1$, and $u = pa + qb$ is in the rectangle, not on the far edges. If $v$ is in $L$, then $v$ can’t be on the near edges of the rectangle either, so it is in interior. □
Lemma 2. $L$ is either $L_1$ or $L_2$.

proof. We note that $b = 2c - a$ is in $L_2$. Therefore $L_1 \subset L_2$, and since $a$ and $b$ are in $L$, $L_1 \subset L$.

Suppose that $L$ contains an element $v$ not in $L_1$. We write $v = w + u$ as in the previous lemma, with $u = (u_1, u_2)^t$ in the interior of the rectangle $0, a, b, a + b$. So $0 < u_1 < a$ and $0 < u_2 < b_2$. Then $u + ru = (2u_1, 0)^t$ is in $L$, and since it is horizontal, $u + ru$ is an integer multiple of $a$. But $0 < 2u_1 < 2a_1$.

The only possibility is that $u_1 = \frac{1}{2}a_1$. Similarly, $u + ru = (0, u_2)^t$ is in $L$, and $u_2 = \frac{1}{2}b_2$. So $u = \frac{1}{2}(a + b) = c$. So if $L_1 < L$, then $L = L_2$. □

The reflections and glides in $G$.

We ask: Are the reflections $\tau$ and $\bar{\sigma}$ of $\bar{G}$ the images of reflections in $G$? If so, we can put the origin at the intersection of the lines of reflection. Then $\tau$ and $\bar{\sigma}$ will be in $G$, and we will be happy.

To start, we put the origin at a point that has a rotation by $\pi$ in $G$. Then $\rho = \rho_\pi$ is in $G$.

Lemma 3. Let $v = (v_1, v_2)^t$ be a vector, and let $z = \frac{1}{2}v_2$. The isometry $g = t_{v}r$ is either a reflection or a glide. The horizontal line $\ell : \{x = z\}$ is the line of reflection or the glide line, and $g$ is a reflection about $\ell$ if and only if $v$ is vertical: $v = (0, 2z)^t$.

proof. Since $g$ reverses orientation, it is a reflection or a glide. It suffices to show that $g$ carries the line $\ell$ to itself. The next computation does this. Let $x = (x_1, z)^t$ be a point of the line $\ell$.

$$g(x) = t_{v}r(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ z \end{pmatrix} + \begin{pmatrix} v_1 \\ 2z \end{pmatrix} = \begin{pmatrix} x_1 + v_1 \\ z \end{pmatrix}$$

Since $\tau$ is in the point group, $G$ must contain an element $g = t_{v}r$ that maps to $\tau$, though we don’t know whether or not the translation $t_v$ by itself is an element of $G$.

We can multiply $g$ on the left by any element $t_w$ with $w \in L$. The result $t_{w+v}r$ will be another element whose image in $\bar{G}$ is $\tau$. We write $v = w + u$ as in Lemma 1. Then $t_{-w}t_{v}r = t_{u}r$ is an element of $G$ that maps to $\tau$, and $u = pa + qb = (pa_1, qb_2)^t$ with $0 \leq p, q < 1$. We relabel $t_{u}r$ as $g$.

The element $g^2 = (t_{u}r)(t_{u}r) = t_{u}t_{u}rr = t_{u+ru}$ is in $G$, and therefore $u + ru = (2pa_1, 0)^t$ is in $L$. It is an integer multiple of $a$. Since $0 \leq p < 1$, $u + ru$ is either $0$ or $a$, and then $u_1$ will be $0$ or $\frac{1}{2}a_1$.

Since $t_{u}r$ and $\rho$ are in $G$, so is $t_{u}r\rho = t_{u}s$. Then $u + \bar{\sigma}u$ will be a vertical vector, either $0$ or $b$, and so $u_2$ can be $0$ or $\frac{1}{2}b_1$. The four possibilities $u + \bar{\sigma}u = 0$ or $a$ and $u + \bar{\sigma}u = 0$ or $b$ show that $u$ is one of the four vectors $0, \frac{1}{2}a, \frac{1}{2}b$, or $\frac{1}{2}(a + b) = c$.

Suppose that $L = L_2$. Then we eliminate the fourth possibility $u = c$ because if $t_{u}r$ is in $G$, so is $r = t_{0}r$. Next, if $u$ is the vertical vector $\frac{1}{2}b$, then $t_{u}r$ is a reflection about some horizontal axis. So $\tau$ is represented by a reflection. Also, $t_{-u}s$ will be in $G$, and $-c + u$ is a horizontal vector. Therefore $\tau$ is represented by a reflection too. This shows that there is only one type of group, when $L = L_2$.

Suppose that $L = L_1$. The two possibilities $\frac{1}{2}a$ and $\frac{1}{2}b$ are interchanged when we switch axes, so we can eliminate one of them. This leaves at most three possibilities for $G$ when $L = L_1$, making four possibilities in all. Table (6.6.2) of the text confirms that there are (at least) four patterns with point group $D_2$. Reading the table in the usual order, the first is the pattern of lozenges. The second brick pattern is the one with translation group $L_2$. 