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MIT (Fall 2020)

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Due Thu December 3, 2020.
Turn it in via the canvas course website.

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1 Bifurcations of a Critical Point for a 1-D map

Statement: Bifurcations of a Critical Point for a 1D map

When studying bifurcations of critical points in the lectures, we argued that we could understand most of the relevant possibilities simply by looking at 1-D flows. The argument was based on the idea that, generally, a stable critical point will become unstable in only one direction at a bifurcation — so that the flow will be trivial in all the other directions, and we need to concentrate only on what occurs in the unstable direction. The only important situation that is missed by this argument, is the case when two complex conjugate eigenvalues become unstable (Hopf bifurcation). Because in real valued systems eigenvalues arise either in complex conjugate pairs or as single real ones, the cases where only one (or a complex pair) go unstable are the the only ones likely to occur (barring situations with special symmetries that "lock" eigenvalues into synchronous behavior).
The same argument can be made when studying bifurcations of limit cycles in any number of dimensions. In this case one considers the Poincaré map near the limit cycle,[1] with the role of the eigenvalues taken over by the Floquet multipliers. Again, we argue that we can understand a good deal of what happens by replacing the (multi-dimensional) Poincaré map by a one dimensional map with a stable fixed point, and asking what can happen if the fixed point changes stability as some parameter is varied. In this problem you will be asked to do this.

Remark 1.1 *Some important cases are missed by this approach: the case where a pair of complex Floquet multipliers becomes unstable (Hopf bifurcation of a limit cycle), and the cases where a bifurcation occurs because of an interaction of the limit cycle with some other object (e.g., a critical point). Several examples of these situations can be found in section 8.4 of Strogatz' book (Global Bifurcations of Cycles).* ♣

[1] The limit cycle is a fixed point for this map.

Consider a one dimensional (smooth) map from the real line to itself
$$x \longrightarrow y = f(x,\, \mu) \tag{1.1}$$
that depends on some (real valued) parameter μ. **Assume that $x = 0$ is a fixed point for all values of μ — that is, $f(0, \mu) \equiv 0$.** Furthermore, assume that $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$. That is:

$$\left|\frac{\partial f}{\partial x}(0,\, \mu)\right| \;<\; 1 \quad \text{for } \mu < 0, \quad \text{and} \tag{1.2}$$

$$\left|\frac{\partial f}{\partial x}(0,\, \mu)\right| \;>\; 1 \quad \text{for } \mu > 0. \tag{1.3}$$

A further assumption, that involves no loss of generality (since the parameter μ can always be re-defined to make it true) is that
$$\frac{\partial^2 f}{\partial x \partial \mu}(0,\, 0) \neq 0. \tag{1.4}$$
This guarantees that the loss of stability is linear in μ, as μ crosses zero. This is what is called a *transversality condition.* It means this:

> Graph of the Floquet multiplier $\dfrac{\partial f}{\partial x}(0, \mu)$ as a function of μ. Then the resulting curve crosses one of the lines $y = \pm 1$ transversally (curves not tangent at the common point) for $\mu = 0$.

By doing an appropriate expansion of the map f for x and μ small (or by any other means), **show that** (generally[2]) **the following happens:**

a. ***For $\dfrac{\partial f}{\partial x}(0,\, 0) = 1$, either:***

- **a1.** *Transcritical bifurcation (no special symmetries assumed for f):* There exists another fixed point, $x_* = x_*(\mu) = O(\mu)$, such that: $x_* \neq 0$ is unstable for $\mu < 0$ and $x_* \neq 0$ is stable for $\mu > 0$. The two points "collide" at $\mu = 0$ and exchange stability.
- **a2.** *Supercritical or soft pitchfork bifurcation, assuming that f is an odd function of x:* Two stable fixed points exist for $\mu > 0$, one on each side of $x = 0$, at a distance $O(\sqrt{\mu})$. All three points merge for $\mu = 0$.
- **a3.** *Subcritical or hard pitchfork bifurcation, assuming that f is an odd function of x:* Two unstable fixed points exist for $\mu < 0$, one on each side of $x = 0$, at a distance $O(\sqrt{-\mu})$. All three points merge for $\mu = 0$.

What does all this mean in the context of the Poincaré map for a limit cycle?

b. ***For $\dfrac{\partial f}{\partial x}(0,\, 0) = -1$ (no special symmetries assumed for f), either:***

- **b1.** *Supercritical or soft flip bifurcation:* For $\mu > 0$ two points $x_1(\mu) \approx -x_2(\mu) = O(\sqrt{\mu})$ exist, on each side of the fixed point $x = 0$, with $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1,\, x_2\}$ is a period two orbit for the map (1.1). **Show that this orbit is stable.**
- **b2.** *Subcritical or hard flip bifurcation:* For $\mu < 0$ two points $x_1(\mu) \approx -x_2(\mu) = O(\sqrt{-\mu})$ exist, on each side of the fixed point $x = 0$, with $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1,\, x_2\}$ is a period two orbit for the map (1.1). **Show that this orbit is unstable.**

In the context of the Poincaré map for a limit cycle, a flip bifurcation corresponds to a period doubling bifurcation of the limit cycle.

Hint 1.1 *If you expand f in a Taylor expansion near $x = 0$ and $\mu = 0$, up to the leading order beyond the trivial first term ($f \sim \pm x$), and you make sure to keep* ***all*** *the relevant terms (and nothing else), and you make sure to identify certain terms that must vanish, the problem reduces to (mostly) trivial high-school algebra. In figuring out what to*

[2] There are special conditions under which all this fails. You must find them as part of your analysis. What are they?

keep and what not to keep, note that you have two small quantities (x and μ), whose sizes are related. The process is very similar to the Hopf bifurcation expansion calculation (e.g. see course notes), but much simpler computationally.

For part (b) you will have to keep in the expansion not just the leading order terms beyond the trivial first term, but the next order as well. *Reason: In this case, you will be looking for solutions to the equation $f(f(x, \mu), \mu) = x$. But, when you calculate $f(f(x, \mu), \mu)$, you will see that the second order terms cancel out — thus the need for an extra term in the expansion. This is the same phenomena that forces the Hopf bifurcation calculation to third order.* ♣

IMPORTANT

To standardize the notation, **use the following symbols in your answer:**

$$\boldsymbol{\nu = \frac{\partial f}{\partial x}(0,\, 0)} \quad \text{— note that } \nu = \pm 1,$$

$$\boldsymbol{a = \frac{1}{2}\,\frac{\partial^2 f}{\partial x^2}(0,\, 0)}, \qquad \boldsymbol{b = \frac{\partial^2 f}{\partial x\, \partial \mu}(0,\, 0)}, \qquad \boldsymbol{c = \frac{1}{6}\,\frac{\partial^3 f}{\partial x^3}(0,\, 0)}, \qquad \text{and} \qquad \boldsymbol{d = \frac{1}{2}\,\frac{\partial^3 f}{\partial x^2\, \partial \mu}(0,\, 0)}.$$

Then *obtain leading order expressions for the fixed points and flip-bifurcation orbits in terms of these quantities.*

2 Sierpinski gasket

Statement: Sierpinski gasket

Consider the fractal (a "Sierpinski gasket") in the plane, made in the following **recursive fashion:**

1. Start with an equilateral triangle, with sides of length L.
2. Draw the lines joining the sides mid-points, and divide it into four equal equilateral sub-triangles.
3. Remove the sub-triangle at the center.
4. Repeat the process with each of the other three remaining sub-triangles.

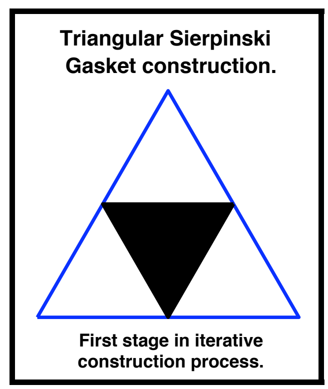

Figure 2.1: The picture on the right illustrates the recursion, showing the result of the first iteration in the process described above.

Now, do the following:

A. Calculate the box dimension of the fractal.

B. Calculate the self-similar dimension of the fractal.

C. Calculate the surface area of the fractal.

D. Show that the fractal has as many points as a full square — **This part is hard(er).**

E. Let d_s be the dimension calculated in part **A**. Modify the construction of the fractal, in such a way that the modified fractal can be selected to have **any** given box dimension $0 < d < d_s$.
Hint: take out bigger chunks at each stage.

F. Construct fractals (subsets of the plane) such that their box dimensions can be selected to have **any** given box dimension $d_s < d < 2$.

3 Problem 08.06.03 - Strogatz (Irrational flow yields dense orbits)

Statement for problem 08.06.03

Consider the flow in the torus given by

$$\frac{d\theta_1}{dt} = \omega_1 \quad \text{and} \quad \frac{d\theta_2}{dt} = \omega_2, \tag{3.1}$$

where ω_1 and ω_2 are non-zero constants, θ_1 and θ_2 are angles (thus we identify any two values that differ by a multiple of 2π) and $\mu = \omega_1/\omega_2$ is *irrational*. Show that each trajectory is **dense**; i.e., given any point p on the torus, any initial condition q, and any $\epsilon > 0$, there is some $t < \infty$ such that the trajectory starting at q passes within a distance ϵ of p.

Hint #1. The lemma below provides a key step in the argument. **Prove it.**

Lemma: *Let $\mathcal{Z}$ be the set of all the integers and let μ be an irrational number. Then $2\,\pi\,\mu\,\mathcal{Z}\ (\mathrm{mod}\,2\,\pi)$ is dense in $[0,\,2\,\pi)$.*

Note: Two numbers are equal mod $2\,\pi$ if their difference is an integer multiple of $2\,\pi$ — i.e.: think of the numbers as angles in radians. In particular, think of $2\,\pi\,\mu\,\mathcal{Z}\ (\mathrm{mod}\,2\,\pi)$ as the set of all the numbers $0 \le x < 2\,\pi$ which are equal mod $2\,\pi$ to a number of the form $2\,\pi\,\mu\,n$, with $n \in \mathcal{Z}$.

Hint #2. Show that the numbers in $2\,\pi\,\mu\,\mathcal{Z}\ (\mathrm{mod}\,2\,\pi)$ are all different. Since they are all in $0 \le x < 2\,\pi$, what happens with the distances between them? Exploit this and the two equalities: $x_p - x_q = x_{p-q}$ and $x_{p+jk} = x_p + j\,x_k$, which (of course) also apply $\mathrm{mod}\,2\,\pi$.

Hint #3. Identify the Torus with the rectangle $\mathcal{R} = [0,\,2\,\pi) \times [0,\,2\,\pi)$, and (for any orbit) consider the set produced by its intersections with the bottom side of $\mathcal{R}$ — that is $\theta_2 = 0\ (\mathrm{mod}\,2\,\pi)$.

4 Problem 08.06.05 - Strogatz (Plotting Lissajous figures)

Statement for problem 08.06.05

Using a computer, plot the curve whose parametric equations are $x(t) = \sin(t)$ and $y(t) = \sin(\omega\,t)$, for the following rational and irrational values of the parameter ω.

$$\left.\begin{array}{lll} \textbf{(a)}\ \omega = 3 & \textbf{(c)}\ \omega = \dfrac{2}{3} & \textbf{(c)}\ \omega = \dfrac{5}{3} \\ \textbf{(b)}\ \omega = \sqrt{2} & \textbf{(d)}\ \omega = \pi & \textbf{(f)}\ \omega = \dfrac{1}{2}\left(1+\sqrt{5}\right) \end{array}\right\} \tag{4.1}$$

The resulting curves are called *Lissajous* figures. They can be displayed on an oscilloscope by using two ac signals of different frequencies as inputs.

Remark 4.2 *Obviously, in a numerical calculation ω cannot be "irrational". However, an irrational ω will be approximated by a rational number which is the quotient of two very large integers. The result will be a curve with a very, very, complicated and a very, very long period.*

5 Problem 09.02.02 - Strogatz. Ellipsoidal trapping region for the Lorenz attractor

Statement for problem 09.02.02

Consider the Lorenz system of equations:

$$\frac{dx}{dt} = \sigma\,(y - x), \quad \frac{dy}{dt} = r\,x - y - x\,z, \qquad \text{and} \qquad \frac{dz}{dt} = x\,y - b\,z, \tag{5.1}$$

where σ, r and b are **positive** constants. Let Ω be the ellipsoidal region given by the points satisfying the equation

$$E = r\,x^2 + \sigma\,y^2 + \sigma\,(z - 2\,r)^2 \leq C, \tag{5.2}$$

where $C > 0$ is a constant. **Show that,** for C large enough, all trajectories of the Lorenz equations eventually enter Ω and stay there forever. *Hint: compute* $\boldsymbol{\frac{dE}{dt}}$.

For a much stiffer challenge: try to obtain the smallest possible value of C with this property.

6 Problem 09.06.02 - Strogatz (Pecora and Carroll's approach)

Statement for problem 09.06.02

Pecora and Carroll's approach for signal transmission/reception using the Lorenz system. In the pioneering work of Pecora and Carroll[3] one of the receiver variables is simply set *equal* to the corresponding transmitter variable. For instance, if $x(t)$ is used as the transmitter drive signal, then the receiver equations are

$$\left.\begin{aligned} x_r(t) &\equiv x(t), \\ \frac{dy_r}{dt} &= r\,x(t) - y_r - x(t)\,z_r, \\ \frac{dz_r}{dt} &= x(t)\,y_r - b\,z_r, \end{aligned}\right\} \tag{6.1}$$

where the first equation is **not** a differential equation.[4] Their numerical simulations, and a heuristic argument, suggested that $y_r(t) \to y(t)$ and $z_r(t) \to z(t)$ as $t \to \infty$, even if there were differences in the initial conditions.

Here are the steps for simple proof of the result stated above, due to He and Vaidya.[5]

[3] Pecora, L. M., and Carroll, T. L., *Synchronization in chaotic systems.* Phys. Rev. Lett. **64:**821, (1990).

[4] This equation replaces the first equation $\dot{x_r} = \sigma\,(y_r - x_r)$ in a Lorenz system for $(x_r,\, y_r,\, z_r)$. Then x is used to replace x_r in the other two equations. The Lorenz system constants are σ, r, b.

[5] He, R., and Vaidya, P. G., *Analysis and synthesis of synchronous periodic and chaotic systems.* Phys. Rev. A, **46:**7387 (1992).

A. Show that the error dynamics are governed by:

$$\left.\begin{aligned} e_x(t) &\equiv 0, \\ \frac{de_y}{dt} &= -e_y - x(t)\, e_z, \\ \frac{de_z}{dt} &= x(t)\, e_y - b\, e_z, \end{aligned}\right\} \tag{6.2}$$

where $\boldsymbol{e_x = x - x_r}$, $\boldsymbol{e_y = y - y_r}$, and $\boldsymbol{e_z = z - z_r}$.

B. Show that $\boldsymbol{V = (e_y)^2 + (e_z)^2}$ is a Liapunov function.

C. What do you conclude?

7 Problem 14.10.03 - Newton's method in the complex plane

Statement for problem 14.10.03

Suppose that you want to solve an equation, $g(x) = 0$. Then you can use *Newton's method*, which is as follows: Assume that you have a "reasonable" guess, x_0, for the value of a root. Then the sequence $\boldsymbol{x_{n+1} = f(x_n)}$, $\boldsymbol{n \geq 0}$, where converges (very fast) to the root.

$$\boldsymbol{f(x) = x - \frac{g(x)}{g'(x)}}, \tag{7.1}$$

Remark 7.3 (The idea). *Assume an approximate solution $g(x_a) \approx 0$. Write $x_b = x_a + \delta x$ to improve it, where δx is small. Then $0 = g(x_a + \delta x) \approx g(x_a) + g'(x_a)\, \delta x \;\Rightarrow\; \delta\, x \approx -\frac{g(x_a)}{g'(x_a)}$, and (7.1) follows.*
Of course, if x_0 is not close to a root, the method may not converge. Even if it converges, it may converge to a root that is far away from x_0, not necessarily the closest root. In this problem we investigate the behavior of Newton's method in the complex plane, *for arbitrary starting points.*

Consider iterations of the map in the complex plane generated by Newton's method for the roots of $z^3 - 1 = 0$. That is

$$z_{n+1} = f(z_n) = \left(\frac{2}{3} + \frac{1}{3\, z_n^3}\right) z_n, \quad n \geq 0, \tag{7.2}$$

where $0 < |z_0| < \infty$ is arbitrary. Note that

$$\boldsymbol{\zeta_1 = 1, \quad \zeta_2 = e^{i\, 2\, \pi/3} = \frac{1}{2}\,(-1 + i\,\sqrt{3}),} \quad \text{and} \quad \boldsymbol{\zeta_3 = e^{i\, 4\, \pi/3} = \frac{1}{2}\,(-1 - i\,\sqrt{3}),} \tag{7.3}$$

are the roots of $z^3 = 1$.

Your tasks: Write a computer program to calculate the orbits $\{z_n\}_{n=0}^{\infty}$. Then, for every [6] initial point z_0, draw a colored dot at the position of z_0, where the colors are picked as follows:

$z_n \to \zeta_1$, cyan. $\quad z_n \to \zeta_2$, magenta. $\quad z_n \to \zeta_3$, yellow. $\quad$ No convergence, black. (7.4)

What do you see? Do blow ups of the limit regions between zones.

Hint. Deciding that the sequence converges is easy: once z_n gets "close enough" to one of the roots, then the very design of Newton's method guarantees convergence. Thus, given a z_0, compute z_N for some large N, and check if $|z_N - \zeta_j| < \delta$ for one of the roots and some "small" tolerance δ — which does not have to be very small, in fact $\delta = 0.25$ is good enough. You can

[6] Numerically this means: choose a sufficiently fine grid in a rectangle, and pick every point in the grid. For example, select the square $-2 < x < 2$ and $-2 < y < 2$, where $z_0 = x + i\, y$.

get pretty good pictures with $N = 50$ iterations on a 150×150 grid. A larger N is needed when refining near the boundary between zones.

Hint. If you use MatLab, do not plot "points". Instead, plot "regions", where the color of each pixel is decided by z_0 — use the command image$(x,\, y,\, C)$ *to plot. Why? Because using points leaves a lot of unpainted space in the figure, and gives much larger file sizes.*

8 Problem 11.03.08 - Strogatz. Sierpinski's carpet

Statement for problem 11.03.08

Consider the process shown in figure 8.1. The closed unit box is divided into nine equal boxes, and the open central box is deleted. Then this process is repeated for each of the eight remaining sub-boxes, and so on. Figure 8.1 shows the first two stages.

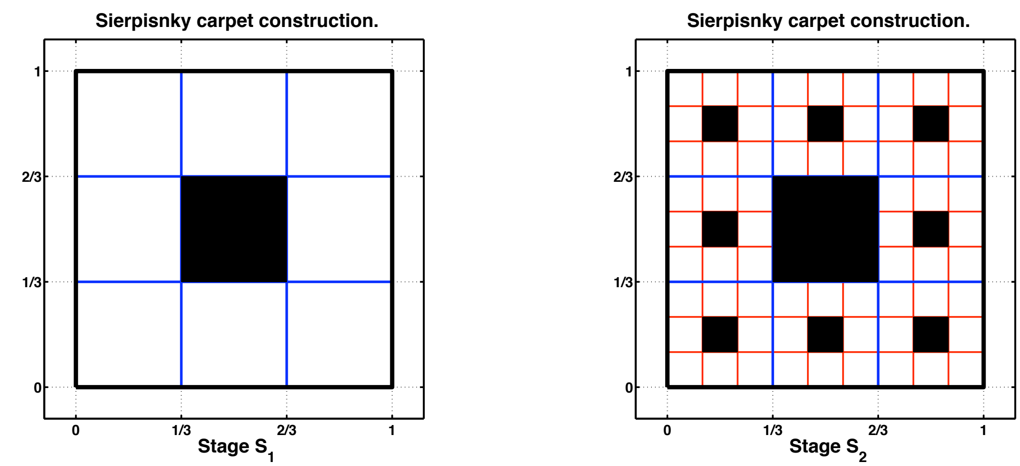

Figure 8.1: Problem 11.3.8. **Sierpinski carpet construction.** The areas shaded in black are the parts of the original square deleted at each stage of the fractal's construction.

A. Sketch the next stage, S_3.

B. Find the similarity dimension of the limiting fractal, known as the **Sierpinski carpet.**

C. Show that the Sierpinski carpet has zero area.

THE END.