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MIT (Fall 2020)

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Due Friday November 13, 2020.
Turn it in via the canvas course website.

Contents

1 Large μ limit for Liénard system #03

Statement: Large μ limit for Liénard system #03

A Liénard equation has the form
$$\ddot{x} + \mu\, f'(x)\,\dot{x} + g(x) = 0, \tag{1.1}$$
for some functions f and g. Here $\mu > 0$ is a parameter.

This can be re-written in the form
$$\frac{d}{dt}\left(\dot{x} + \mu\, f(x)\right) + g(x) = 0. \tag{1.2}$$
Introduce $y = \frac{1}{\mu}\,\dot{x} + f(x)$, to get the system
$$\dot{x} = \mu\,(y - f(x)) \quad \text{and} \quad \dot{y} = -\frac{1}{\mu}\,g(x). \tag{1.3}$$

In this problem we will consider the case
$$f(x) = -x + \frac{1}{3}\,x^3 - \frac{1}{60}\,x^5 \quad \text{and} \quad g(x) = x, \tag{1.4}$$
with $\boldsymbol{\mu \gg 1}$.

Analyze the large μ limit for this system. In particular:

1. Are there any limit cycles? Are they stable, unstable, semi-stable?

2. Are there any critical points? Are they attractors, repellers?

3. Does the system have any global attractor?

4. Sketch the phase plane portrait.

2 Phase Plane Surgery #01

Statement: Phase Plane Surgery #01

Can a smooth vector field exist in the plane such that:

— The critical points are $P_1 = (-2, 0)$, $P_2 = (0, 0)$ and $P_3 = (2, 0)$.

— All the critical points are spirals.

— The circles with radii: $R_1 = 1$ centered at P_1, $R_2 = 4$ centered at P_2, and $R_3 = 1$ centered at P_3, are orbits.

Would your answer change if P_2 is a saddle?

In either case, if your answer is yes, sketch the way the orbits might look in an example satisfying the criteria above.

Challenge question: In either case, if your answer is yes, can you give an actual example (i.e.: write the vector field explicitly) that gives you a phase portrait with the same qualitative features (the closed orbits need not be circles for this).

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3 Simple Poincaré Map for a limit cycle #02

Statement: Simple Poincaré Map for a limit cycle #02

Consider the following autonomous phase plane system

$$\left.\begin{aligned} \frac{dx}{dt} &= (x^2 + y^4)\left(\nu\, x - \tfrac{\nu}{4}\, x^3 - x^2\, y - \nu\, x\, y^2 - 4\, y^3\right), \\ \frac{dy}{dt} &= (x^2 + y^4)\left(\nu\, y + \tfrac{1}{4}\, x^3 - \tfrac{\nu}{4}\, x^2\, y + x\, y^2 - \nu\, y^3\right), \end{aligned}\right\} \quad \text{where } \nu > 0. \tag{3.1}$$

This system has a periodic solution **(show this)**, which can be written in the form

$$x = 2\cos\Phi, \quad y = \sin\Phi, \quad \text{where} \quad \frac{d\Phi}{dt} = 2\,(x^2 + y^4) = 2\,(1 + \cos^2\Phi)^2. \tag{3.2}$$

This solution produces an orbit going through the point $x = 0$, $y = 1$ in the phase plane. The orbit is an ellipse, as (3.2) shows. [1]

Construct (either numerically [2] or analytically) a Poincaré map near this orbit, and use it to show that the orbit is a stable limit cycle. *Define the Poincaré map* $\boldsymbol{z} \to \boldsymbol{u} = \boldsymbol{P}(\boldsymbol{z})$ as follows:

- For every sufficiently small $\boldsymbol{z}$, let $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{t},\, \boldsymbol{z})$ and $\boldsymbol{y} = \boldsymbol{Y}(\boldsymbol{t},\, \boldsymbol{z})$ be the solution of (3.1) defined by $\boldsymbol{X}(\boldsymbol{0},\, \boldsymbol{z}) = \boldsymbol{0}$ and $\boldsymbol{Y}(\boldsymbol{0},\, \boldsymbol{z}) = \boldsymbol{1} + \boldsymbol{z}$.
- For this solution the polar angle $\boldsymbol{\theta}$ in the phase plane is an *increasing function of time*, starting at $\boldsymbol{\theta} = \frac{1}{2}\boldsymbol{\pi}$ for $\boldsymbol{t} = \boldsymbol{0}$. Thus, there is a time $\boldsymbol{t} = \boldsymbol{t_z}$ at which the solution reaches $\boldsymbol{\theta} = \frac{5}{2}\boldsymbol{\pi}$ (note that $\boldsymbol{t_z}$ is a function of $\boldsymbol{z}$). Then take $\boldsymbol{u} = \boldsymbol{Y}(\boldsymbol{t_z},\, \boldsymbol{z}) - \boldsymbol{1}$.

Hint. Because t_z is a function of z, unknown a priori, the definition of the Poincaré map above is a bit awkward to implement. To avoid having to calculate t_z for each solution, it is a good idea to use a parameter other than time to describe the orbits. For example, if the equations are written in terms of a parameter such as the polar angle — namely $\frac{d\boldsymbol{x}}{d\boldsymbol{\theta}} = \boldsymbol{F}(\boldsymbol{x},\, \boldsymbol{y})$ and $\frac{d\boldsymbol{y}}{d\boldsymbol{\theta}} = \boldsymbol{G}(\boldsymbol{x},\, \boldsymbol{y})$, then the Poincaré map is easier to describe, as θ varies from $\theta = \frac{1}{2}\pi$ to $\theta = \frac{5}{2}\pi$ in every one of the orbits needed to compute $u = P(z)$. Note that this is just a "for example", using the polar angle is not the best choice. *Scale the variables first, so that the limit circle is a circle, not an ellipse.* ♣

Small challenge: You should be able to write $\boldsymbol{P}$ analytically. The formula is not even messy.

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[1] Note that $\boldsymbol{\Phi}$ is a strictly increasing function of time.

[2] If you do it numerically, keep ν as a variable and check your answers for several values — say: $\nu = 0.1,\ 0.5,\ 1,\ 2,\ 5$.

4 Problem 07.02.x1 (area evolution)

Statement for problem 07.02.x1

Consider some arbitrary orbit, Γ, for the phase plane system

$$\frac{d\vec{r}}{dt} = \vec{F}(\vec{r}) \quad \text{where} \quad \vec{r} = (x,\, y)^T, \quad \vec{F} = (f(x,\, y),\, g(x,\, y))^T, \tag{4.1}$$

and $\vec{F}$ has continuous partial derivatives up to (at least) second order. That is: Γ is the curve in the plane given by some solution $\vec{r} = \vec{r}_\gamma(t)$ to (4.1). Then

A. Let $\Omega = \Omega(t)$ be an "infinitesimal" region that is being advected, along Γ, by the flow given by (4.1). For example:

  A1. Let $\Omega(0)$ be a disk of "infinitesimal" radius dr, centered at $\vec{r}_\gamma(0)$.

  A2. For every point $\vec{r}_p^{\,0} \in \Omega(0)$, let $\vec{r} = \vec{r}_p(t)$ be the solution to (4.1) defined by the initial data $\vec{r}_p(0) = \vec{r}_p^{\,0}$.

  A3. Then, at any time t_*, the set $\Omega(t_*)$ is given by all the points $\vec{r}_p(t_*)$, where $\vec{r}_p^{\,0}$ runs over all the points in $\Omega(0)$.

  Note that $\Omega(0)$ need not be a disk. Any infinitesimal region containing $\vec{r}_\gamma(0)$ will do. All we need is that the notion of area applies to it — see item **B**.

B. Let $\mathcal{A} = \mathcal{A}(t)$ be the area of $\Omega(t)$.

Find a differential equation for the time evolution of $\mathcal{A}$. *The equation that you will find is trivially extended to higher dimensions — e.g. to characterize the evolution of the volume in a 3-D phase space.*

Hints.

h1. First, introduce the vector $\delta\vec{r} = \delta\vec{r}(t) = \vec{r}_p - \vec{r}_\gamma$ for every point in $\Omega(t)$. This vector characterizes the evolution of the "shape" of Ω as the set moves along Γ. In order to calculate how $\mathcal{A}(t)$ evolves, you only need to know how the $\delta\vec{r}$ vectors evolve.

h2. For every vector $\delta\vec{r}$, write an equation giving $\delta\vec{r}(t + dt)$ in terms of $\delta\vec{r}(t)$ and the partial derivatives of $\vec{F}$ along Γ. Since you are dealing with infinitesimal terms, you can neglect higher order terms, so as to obtain a relationship from $\delta\vec{r}(t)$ to $\delta\vec{r}(t + dt)$ given by a linear transformation. Make sure that this linear transformation correctly includes the $O(dt)$ terms, which you will need to calculate time derivatives.

h3. From the transformation in item **h2** derive a relationship between $\mathcal{A}(t + dt)$ and $\mathcal{A}(t)$ — use the fact that, for linear transformations, areas are related by the absolute value of the determinant. You need to calculate the determinant only up to $O(dt)$.

h4. Use the result in item **h3** to calculate the time derivative of $\mathcal{A}$, and obtain the differential equation.

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5 Problem 07.05.06 - Strogatz (Biased van der Pol)

Statement for problem 07.05.06

Suppose the van der Pol oscillator is biased by a constant force:

$$\frac{d^2x}{dt^2} + \mu\,(x^2 - 1)\frac{dx}{dt} + x = a, \tag{5.1}$$

where a can be positive, negative, or zero. (Assume $\mu > 0$ as usual).

a) Find and classify all the fixed points.

b) Plot the nullclines in the Liénard plane. Show that if they intersect on the *middle* branch of the cubic nullcline, the corresponding fixed point is unstable.

c) For $\mu \gg 1$, show that the system has a stable limit cycle if and only if $|a| < a_c$, where a_c is to be determined. (*Hint. Use the Liénard plane.*)

d) Sketch the phase portrait for a slightly greater than a_c. Show that the system is *excitable* — it has a globally attracting fixed point, but some (small, but not infinitesimal) disturbances can send the system on a long excursion through phase space before returning to the fixed point; compare with Exercise 4.5.3.

This system is closely related to the Fitzhugh-Nagumo model of neural activity; for an introduction see Murray, J. (1989) *Mathematical Biology* (Springer, New York) or Edelstein-Keshet, L. (1988) *Mathematical Models in Biology* (Random House, New York).

6 Problem 08.02.05 - Strogatz (Hopf bifurcation using a computer)

Statement for problem 08.02.05

For the following system

$$\frac{dx}{dt} = y + \mu\, x \qquad \text{and} \qquad \frac{dy}{dt} = -x + \mu\, y - x^2\, y, \tag{6.1}$$

a Hopf bifurcation occurs at the origin when $\mu = 0$. Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of μ, verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius R scales with μ as predicted by theory.

7 Problem 08.02.07 - Strogatz (Hopf and homoclinic bifurcations using a computer)

Statement for problem 08.02.07

For the following system

$$\frac{dx}{dt} = \mu\, x + y - x^2 \qquad \text{and} \qquad \frac{dy}{dt} = -x + \mu\, y + 2x^2, \tag{7.1}$$

a **Hopf bifurcation occurs** at the origin when $\mu = 0$. Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of μ, verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius R scales with μ as predicted by theory.

In addition to a Hopf bifurcation, this system also exhibits an **homoclinic bifurcation** of the limit cycle. **FIND IT.**

8 Problem 08.04.03 - Strogatz (Homoclinic bifurcation via computer)

Statement for problem 08.04.03

Using numerical integration, find the value of μ at which the system

$$\frac{dx}{dt} = \mu\, x + y - x^2 \qquad \text{and} \qquad \frac{dy}{dt} = -x + \mu\, y + 2\, x^2, \tag{8.1}$$

undergoes a homoclinic bifurcation. Sketch the phase portrait just above and below the bifurcation. In fact:

1. Find and classify all the critical points for all values of $\boldsymbol{\mu}$.
2. For $\mu = 0$ the origin is a center for the linearized equations. What happens for the nonlinear equations? Are the nonlinear terms stabilizing or destabilizing? What sort of critical point is the origin for the full equations: stable spiral, unstable spiral, or center? **You should be able to do this analytically — See hint 8.1.**
3. What happens at $\boldsymbol{\mu}$ crosses $\mathbf{0}$? (Justify your answer). *The result in item 2 should help here!*
4. Increase μ from $\mu = 0$, and find the homoclinic bifurcation (this is where you'll need a computer).
5. **Optional:** Compute the period of the limit cycle as the homoclinic bifurcation is approached, and verify the theoretical prediction: **period** $\boldsymbol{\sim -\log|\mu - \mu_c|}$.

Remark 8.1 *This problem is very similar (same system of equations) to Strogatz problem 8.2.7.* However: 8.2.7 is purely computational, while here you are being asked to do the analysis behind the problem.

Hint 8.1 *To do the analysis in item* **2**, *you have two alternatives:*

A. Do a "two-times expansion" for orbits near the critical point. Namely: write the equations in terms of $x = \epsilon X$ *and* $y = \epsilon Y$ *(where* $0 < \epsilon \ll 1$*). Then expand.*

B. Find a "local Liapunov function", $E = (x^2 + y^2) +$ *higher order terms, such that* $\boldsymbol{\frac{dE}{dt} < 0}$ *near the origin. In fact* $\frac{dE}{dt} \leq 0$ *is O.K., as long as* $\frac{dE}{dt} = 0$ *only for curves the orbits cross — e.g. the axis.*

The first alternative is a straightforward application of the methods in the "Weakly Nonlinear Things" notes. The second actually provides a rigorous proof of the result. However, **it turns out that getting $\boldsymbol{E}$ is not completely trivial!** *The naive approach to searching for E is*

0. Define $E_0 = x^2 + y^2$ *and compute its time derivative. This yields*

$$\frac{dE_0}{dt} = \text{(3rd-order terms)} + \text{(4th-order terms)}.$$

Of course, this is not good enough: the 3rd-order terms can have any sign. Hence:

1. Add 3rd-order term "corrections" to E_0*, to eliminate the 3rd-order terms in* $\dot{E}_0$*. That is, define* $E_1 = E_0 +$ *3rd-order terms, so that*

$$\frac{dE_1}{dt} = \text{(4th-order terms)} + \text{(5th-order terms)}.$$

There is only one way to do this. Unfortunately, some of the 4th-order terms are positive. Hence:

2. Add 4th-order terms "corrections" to E_1*, to eliminate the bad 4th-order terms in* $\dot{E}_1$*. That is, define* $E_2 = E_1 +$ *4th-order terms, so that*

$$\frac{dE_2}{dt} = \text{(negative 4th-order terms)} + \text{(5th-order terms)} + \text{(6th-order terms)}.$$

Again: there is only one way to do this. Unfortunately, this still does not work. Some of the higher order terms here are always smaller than the negative 4-th order terms, but some are not. For example, if $-x^2\,y^2$ *is a negative 4-th order term, then: (i)* $-x^2\,y^2 + x^3\,y^2$ *is always negative for* $x^2 + y^2 \ll 1$*, so* $x^3\,y^2$ *is not a problem, but (ii)* $-x^2\,y^2 + x^4\,y$ *can switch sign (if* $0 < y < x^2 \ll 1$*), so* $x^4\,y$ *is a "bad" term. Hence:*

3. Add 5th-order terms "corrections" to E_2*, to eliminate the bad 5th order terms ... Unfortunately, you then end up with "bad" 6th order terms!*

This never ends! Fortunately: *if you do the process above correctly, you will notice that: while the terms in* E_n *involve ever higher powers of y, there is only a very small set of powers of x that appear. Hence, look for a Liapunov function of the form* $\boldsymbol{E = g(y) + x^2\,f(y) + \ldots}$*, where g, f, etc., are to be determined. This will work: there is a finite (and small) numbers of terms involved. After you have obtained E in this fashion, you will see that it can be expanded as in item B above.*

THE END.