# Problem Set Number 04, 18.385j/2.036j
# MIT (Fall 2020)

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**Due Fri October 30, 2020.**
**Turn it in via the canvas course website.**

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## 1 Problem 14.09.22 (Attracting and Liapunov stable)

### Statement for problem 14.09.22

Recall the *definitions for the various types of stability* that concern critical points:
Let $\mathbf{x}^*$ be a fixed point of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then:

1. $\mathbf{x}^*$ is **attracting** if there is a $\delta > 0$ such that $\lim_{t\to\infty} = \mathbf{x}^*$ whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. That is: any trajectory that starts within $\delta$ of $\mathbf{x}^*$ *eventually* converges to $\mathbf{x}^*$. Note that trajectories that start nearby $\mathbf{x}^*$ *need not stay close in the short run*, but *must approach* $\mathbf{x}^*$ *in the long run.*
2. $\mathbf{x}^*$ is **Liapunov stable** if for each $\epsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for $t > 0$, whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. Thus, trajectories that start within $\delta$ of $\mathbf{x}^*$ stay within $\epsilon$ of $\mathbf{x}^*$ for all $t > 0$.
In contrast with attracting, Liapunov stability requires nearby trajectories to remain close *for all* $t > 0$.
3. $\mathbf{x}^*$ **is asymptotically stable** if it is *both* attracting and Liapunov stable.
4. $\mathbf{x}^*$ is **repeller** if there exist $\epsilon > 0$ and $\delta > 0$ such that: if $0 < \|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then (after some critical time) it will be $\|\mathbf{x}(t) - \mathbf{x}^*\| > \epsilon$ (i.e., for $t > t_c$). Repellers are a special kind of *unstable* critical points.

For each of the following systems, decide whether the origin is attracting but not Liapunov stable, Liapunov stable, asymptotically stable, repeller, or unstable but not a repeller.

a) $\dot{x} = 2\,y$ and $\dot{y} = -3\,x$.

b) $\dot{x} = y\,\cos(x^2+y^2)$ and $\dot{y} = -x\,\cos(x^2+y^2)$.

c) $\dot{x} = -x$ and $\dot{y} = -|y|\,y$.

d) $\dot{x} = 2\,x\,y$ and $\dot{y} = y^2 - x^2$. *Hint: what happens along $x = 0$?*

e) $\dot{x} = x - 2\,y\,x^2 - 4\,y^3$ and $\dot{y} = y + x^3 + 2\,x\,y^2$.

f) $\dot{x} = y$ and $\dot{y} = x$.

g) Finally, consider the critical point $(x,\,y) = (1,\,0)$, for the system

$$\dot{x} = (1-r^2)\,x - (1-\tfrac{x}{r})\,y \quad \text{and} \quad \dot{y} = (1-r^2)\,y + (1-\tfrac{x}{r})\,x, \tag{1.1}$$

defined in the "punctured" plane $r = \sqrt{x^2+y^2} > 0$. *Hint: write the equations in polar coordinates.*

*Additional hints. In some cases you can get the answer by finding a function $\mathcal{J} = \mathcal{J}(x,\,y)$ with a local minimum at the origin such that $\frac{d\mathcal{J}}{dt} > 0$ along trajectories — or maybe one such $\frac{d\mathcal{J}}{dt} < 0$, or maybe one such $\frac{d\mathcal{J}}{dt} = 0$. In other cases look for special trajectories that either leave, or approach, the origin.*

## 2 Problem 06.01.10 - Strogatz (Computer generated phase portrait)

### Statement for problem 06.01.10

First, plot a computer generated phase plane portrait for the "two-eyed monster"

$$\frac{dx}{dt} = y + y^2 \qquad \text{and} \qquad \frac{dy}{dt} = -\frac{1}{2}\,x + \frac{1}{5}\,y - x\,y + \frac{6}{5}\,y^2. \tag{2.1}$$

In particular, make a plot that covers the region $-5 \le x \le 3$ and $-3 \le y \le 2$.
Next find the critical points and classify them. Does what you observe in the plot match what the theory predicts? Explain any discrepancies. *Hint: Explore carefully what happens close to the critical points.*

## 3 Problem 06.02.02 - Strogatz (A trapped solution)

### Statement for problem 06.02.02

Consider the system

$$\left.\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x + (1-x^2-y^2)\,y. \end{aligned}\right\} \tag{3.1}$$

**a.** Let $\boldsymbol{D}$ be the open disk $x^2+y^2 < 4$. Verify that the system satisfies the hypothesis of the existence and uniqueness theorem [1] throughout the domain $\boldsymbol{D}$.

**b.** By substitution, show that $x(t) = \sin(t)$ and $y(t) = \cos(t)$ is an exact solution of the system.

[1] See § 6.2 of Strogatz book — p. 149 (1st edition), p. 150 (2nd edition). A stronger version was stated during Lectures 1-2.

**c.** Now consider a different solution, in this case starting from the initial conditions

$$x(0) = \frac{1}{2} \quad \text{and} \quad y(0) = 0.$$

Without doing any calculations, explain why this solution *must* satisfy $x(t)^2 + y(t)^2 < 1$ for all $t < \infty$.

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# 4 Problem 06.03.11 - Strogatz (A nonlinearity changes a star into a spiral)

## Statement for problem 06.03.11

Here is an example showing that borderline fixed points are sensitive to nonlinear terms. Consider the system in polar coordinates given by

$$\frac{dr}{dt} = -r \qquad \text{and} \qquad \frac{d\theta}{dt} = \frac{1}{\ln(r)}, \quad \text{where } 0 \le r < 1. \tag{4.1}$$

**a)** Write the system in $x$, $y$ coordinates, in the form $\dfrac{dx}{dt} = f(x,\, y)$ and $\dfrac{dy}{dt} = g(x,\, y)$.

**b)** Show that the linearized system about the origin is $\dfrac{dx}{dt} = -x$ and $\dfrac{dy}{dt} = -y$.
Thus the origin is a stable star for the linearization. See remark **4.1**.

**c)** Find $r = r(t)$ and $\theta(t)$ explicitly, given initial conditions $(r_0,\, \theta_0)$ — with $0 < r_0 < 1$.

**d)** Show that the solutions in item **c** satisfy $r(t) \to 0$ and $\theta(t) \to -\infty$ as $t \to \infty$. Therefore the origin is a stable spiral for the nonlinear system.

**Remark 4.1** *This system is not smooth at the origin, since both $f$ and $g$ fail to have second derivatives there. Nevertheless, both have continuous partial derivatives for $r < 1$. Hence, (i) a linearized system at the origin is well defined, (ii) the theorem guaranteeing existence and uniqueness for the solutions applies.*

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# 5 Problem 06.03.13 - Strogatz (A linear center that is a nonlinear spiral)

## Statement for problem 06.03.13

**(Another linear center that is actually a nonlinear spiral).** Consider the system

$$\frac{dx}{dt} = -y - x^3 \qquad \text{and} \qquad \frac{dy}{dt} = x. \tag{5.1}$$

Show that the origin is a spiral, although the linearization predicts a center.

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# 6 Problem 06.05.07 - Strogatz (General relativity and planetary orbits)

## Statement for problem 06.05.07

The relativistic equation for the orbit of a planet around the Sun is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \epsilon\, u^2, \tag{6.1}$$

where $u = 1/r$ and $(r,\, \theta)$ are the polar coordinates for the planet's position in the plane of motion. The parameter $\alpha$ is positive and can be found explicitly from classical Newtonian mechanics. [2] The term $\epsilon\, u^2$ is Einstein's correction, where $\epsilon$ is a very small positive parameter.

**a)** Rewrite the equation as a system in the $(u,\, v)$ plane, where $v = du/d\theta$.

**b)** Find all the equilibrium points of the system.

**c)** Show that one of the equilibria is a center in the $(u,\, v)$ phase plane, according to the linearization. Is it a *nonlinear* center?

**d)** Show that the equilibrium point found in **(c)** corresponds to a circular planetary orbit.

**e)** **Extra questions, not in the book version.** The equation has solutions where $u$ is a periodic function of $\theta$. **(1)** *Do this solutions correspond to periodic orbits around the Sun?* **(2)** *If not, what do they correspond to?* **(3)** *What happens when* $\epsilon = 0$ *(Newtonian mechanics)?*

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# 7 Problem 06.08.09 - Strogatz (Counter-rotating limit cycles)

## Statement for problem 06.08.09

A smooth vector field on the phase plane is known to have exactly two closed trajectories, one of which lies inside the other. The inner circle runs counterclockwise, and the outer runs clockwise.
**True or False:** *there must be at least one fixed point in the region between the cycles.*
**If true, prove it. If false, provide a simple counterexample.**
*Hint: Beware of "gut feeling" instinctive answers. There is a good chance that your intuition is wrong!*

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# 8 Problem 07.02.17 - Strogatz (Dulac's criterion on an annulus)

## Statement for problem 07.02.17

Assume the hypothesis of Dulac's criterion. However, assume that the region of interest, $R$, is (instead of simply connected) topologically equivalent to an annulus (i.e.: it has exactly one hole in it). Using Green's theorem, show that there exists *at most* one closed orbit in $R$ — *this result can be useful as a way of proving that a closed orbit is unique.*

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[2] The parameter $\alpha$ is related to the angular momentum of the orbit.

# 9 Problem 07.03.10 - Strogatz (Existence/non-existence of a limit cycle)

## Statement for problem 07.03.10

Consider the two dimensional system

$$\frac{d\mathbf{x}}{dt} = A\,\mathbf{x} - r^2\,\mathbf{x}, \tag{9.1}$$

where $\boldsymbol{r = \|\mathrm{x}\| = \sqrt{x_1^2 + x_2^2}}$, $A$ is a $2 \times 2$ constant real matrix with complex eigenvalues $\alpha \pm i\,\beta$, and $\beta \neq 0$. Show that there exists at least one periodic orbit if $\alpha > 0$, and that there are none for $\alpha < 0$.

*Hint. Think before you do anything. This is quite easy if you go at it the right way, but if not ...*
*Think of $\dot{r}$ for $r$ large — what can you say about it?*
*Useful vector stuff: for any matrix and vector, $\|A\,\mathbf{x}\| \leq \|A\|\,r$, where $\|A\| \geq 0$ is the norm of the matrix.*
*Useful vector stuff: $|\mathbf{a}\cdot\mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$, where* **the dot $\cdot$ indicates the scalar product.**

# 10 Multiple scales and limit cycles #01

## Statement: Multiple scales and limit cycles #01

Consider the equation

$$\frac{d^2x}{dt^2} - \epsilon\,\cos x\,\frac{dx}{dt} + \frac{1}{\sqrt{\epsilon}}\,\sin(\sqrt{\epsilon}\,x) = 0, \quad \text{where} \quad \boldsymbol{0 < \epsilon \ll 1.} \tag{10.1}$$

Use a multiple scales analysis to calculate the frequency, stability and amplitude of any limit cycle (the frequency up to the first correction beyond linear and the amplitude up to leading order).
*Hint. (i) Using the method in the "Weakly Nonlinear Oscillators" notes, you will find that the leading order approximation has the form $\boldsymbol{x_0 = A(\tau)\,e^{i\,t} + \mathrm{c.c.}}$, and then you will find an equation for $\boldsymbol{A}$ by suppressing resonances. As in the notes, substitute $\boldsymbol{A = \frac{1}{2}\,\rho\,e^{i\,\phi}}$ in this equation, to reduce it to two real valued equations for $\boldsymbol{\rho}$ and $\boldsymbol{\phi}$.*
*(ii) To write the equation for $\boldsymbol{\rho}$ in a compact way, use the equality $\boldsymbol{\pi\,J_1(\rho) = \int_0^{2\pi} \sin(\rho\,\cos(s))\,\cos(s)\,\mathrm{d}s}$, where $\boldsymbol{J_1}$ is the index one Bessel function of the first kind. Because $\boldsymbol{J_1}$ is well studied and understood, this will allow you to conduct the required analysis.*

**THE END.**