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MIT (Fall 2020)

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Due Friday October 16, 2020.
Turn it in via the canvas course website.

Contents

1 Critical points in the phase plane and "energies"

Statement: Critical points in the phase plane and "energies"

Both in the lectures and the book (see example 6.5.1) this claim was made:

$$\textbf{Conservative systems can have neither attracting, nor repelling, fixed points.} \tag{1.1}$$

Consider now the following (linear) systems, with an isolated fixed point at the origin:

$$\left.\begin{aligned} \frac{dx}{dt} &= -x, \\ \frac{dy}{dt} &= a\,y, \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} \frac{dx}{dt} &= -b\,x - y, \\ \frac{dy}{dt} &= -b\,y + x, \end{aligned}\right\} \tag{1.3}$$

where $a \neq 0$, and b are constants.

For the system in (1.2), define E_a by
$$E_a = |x|^a\,y. \tag{1.4}$$

Then, it is easy to check that:
$$\frac{d\boldsymbol{E_a}}{\boldsymbol{dt}} = \mathbf{0} \quad \textbf{along solutions of (1.2).} \tag{1.5}$$

Hence E_a is a conserved quantity. Now, for $a > 0$ the origin in (1.2) is a saddle, so that this last fact is not in contradiction with (1.1). However:

$$\textbf{(1.5) applies even when } \boldsymbol{a < 0} \textbf{ and the origin is an } \underline{\textbf{attracting}} \textbf{ node!} \tag{1.6}$$

Similarly, let:
$$E_b = 2\,b\,\arctan\left(\frac{y}{x}\right) + \ln(x^2 + y^2) = 2\,b\,\theta + \ln(r^2), \tag{1.7}$$

where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then, again, it is easy to check that:

$$\frac{dE_b}{dt} = 0 \text{ for the solutions of (1.3),}$$
$$\text{even when } b \neq 0 \text{ and the origin is an } \underline{\text{attracting/repelling}} \text{ spiral point.} \tag{1.8}$$

Question: Do either of (1.6) or (1.8) contradict (1.1)? Explain.

2 Critical points with zero linear part

Statement: Critical points with zero linear part

Consider the following phase plane systems

$$\dot{x} = \quad (x^2 + y^2)\, y \qquad \text{and} \qquad \dot{y} = -(x^2 + y^2)\, x, \tag{2.1}$$
$$\dot{x} = \quad (x^2 + y^2)\, (y - x/9) \qquad \text{and} \qquad \dot{y} = -(x^2 + y^2)\, (x + y/9), \tag{2.2}$$
$$\dot{x} = -(x^2 + y^2)\, x \qquad \text{and} \qquad \dot{y} = -(x^2 + y^2)\, y/2, \tag{2.3}$$
$$\dot{x} = \quad (x^2 + y^2)\, x \qquad \text{and} \qquad \dot{y} = -(x^2 + y^2)\, y. \tag{2.4}$$

In each of these systems the origin, $\mathcal{O}$, is the only critical point — with linearization matrix $A = 0$. Yet each of these systems has a phase portrait that is entirely analogous to that of a linear system (saddle, center, node, etc.). **Analyze the systems above, classify and sketch their phase plane portraits, and calculate the index for $\mathcal{O}$ in each case.**

Further questions. **(Q1)** For a stable node in a linear system, the solutions approach the critical point exponentially fast as $t \to \infty$. *If one of the systems above has a stable node at the origin, at what rate does $r = \sqrt{x^2 + y^2}$ vanish as t grows?* **(Q2)** For a linear center, the orbital period is independent of the amplitude. *If one of the systems above has a center at the origin, how does the orbital period scale with amplitude?*

3 Index for a center when linearized

Statement: Index for a center when linearized

Consider a phase plane system

$$\dot{x} = f(x,\, y) \quad \text{and} \quad \dot{y} = g(x,\, y), \tag{3.1}$$

where f and g are smooth functions of all of its arguments. Assume that:

1. The origin is a critical point. That is $f(0,\, 0) = g(0,\, 0) = 0$.
2. The origin is a center for the linearized system. That is: if A is the 2×2 matrix corresponding to the linearized system near the critical point at the origin, then A has two complex conjugate, pure imaginary and nonzero, eigenvalues. Equivalently, if $\tau = \text{tr}(A)$ and $\Delta = \det(A)$, then $\mathbf{0 < \Delta}$ and $\boldsymbol{\tau = 0}$.

Because $\Delta \neq 0$, the origin is an isolated critical point (inverse function theorem), and has an index associated with it. Let this index be $\mathcal{I}_0$. **Your task here is to calculate $\mathcal{I}_0$.**

Note. Centers are structurally unstable. Hence *you cannot calculate their index by simply calculating the index for the linearized system,* you have to do a slightly more sophisticated calculation. Interestingly, even though the actual phase plane portrait (when a center occurs for the linearized system) cannot be ascertained from the linearized system alone (structural instability), you need no nonlinear information to calculate the index!

Hint:

3. Write the system (3.1) in vector form

$$\dot{Y} = F(Y), \qquad \text{where } Y = (x,\, y)^T \text{ and } F = (f,\, g)^T. \tag{3.2}$$

where $Y = (x,\, y)^T$ and $F = (f,\, g)^T$.

4. Assume that the matrix A has the form
$$A = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, \tag{3.3}$$

where $\mu > 0$ and $\lambda = \pm i\,\mu$ are the eigenvalues of A. There is no loss of generality here: any matrix with two nonzero, conjugate and purely imaginary eigenvalues, can be reduced to this form by an appropriate choice of coordinates.

5. Consider now the one parameter family of systems

$$\dot{Y} = F(Y) + \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y. \tag{3.4}$$

For these systems the origin is a critical point, with the linearization matrix

$$B = \begin{pmatrix} \epsilon & \mu \\ -\mu & \epsilon \end{pmatrix}. \qquad \text{In particular } \det(B) = \epsilon^2 + \mu^2 > 0, \tag{3.5}$$

so that the origin is an isolated critical point (inverse function theorem).

Let now $\mathcal{I} = \mathcal{I}(\epsilon)$ be the index of the critical point at the origin for (3.4). Calculate $\mathcal{I}(\epsilon)$ for $\epsilon \neq 0$, and use the properties of the index to get $\mathcal{I}(0) = \mathcal{I}_0$.

4 Index for a critical point with zero determinant

Statement: Index for a critical point with zero determinant

Consider a phase plane system
$$\dot{x} = f(x,\, y) \quad \text{and} \quad \dot{y} = g(x,\, y), \tag{4.1}$$

where f and g are smooth functions of all of its arguments. Assume that:

1. The origin $\mathcal{O}$ is an isolated critical point. That is $f(0,\,0) = g(0,\,0) = 0$, and there are no solutions to $f(x,\,y) = g(x,\,y) = 0$ with $0 < x^2 + y^2 < \epsilon$ — for some ϵ.

2. Let A be the 2×2 matrix corresponding to the linearized system near $\mathcal{O}$, with $\tau = \mathrm{tr}(A)$ and $\Delta = \det(A)$. Suppose that $\boldsymbol{\Delta = 0}$ and $\boldsymbol{\tau > 0}$ — so that one eigenvalue of A vanishes, and the other equals τ.

This is a structurally unstable situation, in particular: the index for $\mathcal{O}$ is not determined at all by the linearized equations. **Construct examples of the above situation where:**

A. $\mathcal{I} = \text{index}(\mathcal{O}) = \quad 1.$

B. $\mathcal{I} = \text{index}(\mathcal{O}) = -1.$

C. $\mathcal{I} = \text{index}(\mathcal{O}) = \quad 0.$

Sketch the phase plane portraits for the systems that you construct.

Hints. Consider the linear system $\dot{Y} = A\,Y$, and then add a nonlinear correction which:

For part **A**. Makes $\mathcal{O}$ into a (nonlinear) node.

For part **B**. Makes $\mathcal{O}$ into a (nonlinear) saddle.

For part **C**. Makes $\mathcal{O}$ into a (nonlinear) saddle on one side, and a (nonlinear) node on the other.

5 Index theory - interpolating from saddles to nodes #2

Statement: Index theory - interpolating from saddles to nodes #2

Consider a one parameter family of phase plane systems

$$\dot{x} = f(x,\, y,\, r) \quad \text{and} \quad \dot{y} = g(x,\, y,\, r), \tag{5.1}$$

where f and g are smooth functions of all of its arguments — including the parameter r. Assume that:

1. The origin is a critical point for all values of r. That is $f(0,\, 0,\, r) = g(0,\, 0,\, r) = 0$.

2. For $\boldsymbol{r = 0}$ the origin is an isolated critical point. In fact, a saddle.

3. For $\boldsymbol{r = 1}$ the origin is an isolated critical point. In fact, a node.

Show that there is at least one value $0 < R < 1$, such that: for $r = R$ the origin is not isolated critical point.

Hint. Let $\mathcal{I} = \mathcal{I}(r)$ be the index of the critical point at the origin for (5.1), for any r for which it is defined. Also note that $\mathcal{I}(0) = -1$ and $\mathcal{I}(1) = 1$. Use now the properties of the index.

6 Liapunov Function # 01

Statement: Liapunov Function # 01

Show that the system
$$\frac{dx}{dt} = -x + 2\,y^3 - 2\,y^4 \quad \text{and} \quad \frac{dy}{dt} = -x - y + x\,y, \tag{6.1}$$
has no periodic solutions.

Hint. Find a Liapunov function. Try the form $L = x^m + a\,y^n$.

7 Example of a reversible system that is not conservative

Statement: Example of a reversible system that is not conservative

Give an example of a reversible system that is not conservative.
Hint. *Remember that a phase plane system*

$$\dot{x} = f(x, y) \quad \text{and} \quad \dot{y} = g(x, y), \tag{7.1}$$

is reversible if, for example, f is odd and g is even in y — that is: $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$. In this case the change $t \to -t$ and $y \to -y$ leaves the system invariant.

In addition, we know that conservative systems cannot have sinks or sources. Now, ask yourself: what systems have exactly the opposite property [almost every critical point is either a source or a sink]. Then produce a system of this kind, with f odd and g even.

8 Three limit cycles and enclosed critical points

Statement: Three limit cycles and enclosed critical points

Consider a phase plane system such that, in some region $\boldsymbol{R}$ of the phase plane it has two disjoint limit cycles, both enclosed by a third one — the situation is illustrated in figure 8.1. **For this to be possible: what is the minimum number of critical points that the system needs to have in $\boldsymbol{R}$?**

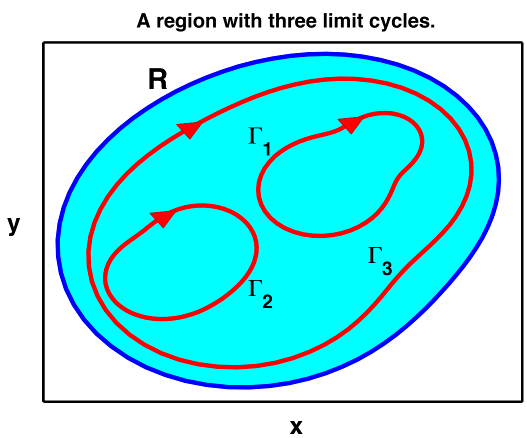

Figure 8.1: Two disjoint limit cycles, both enclosed by a third one, in some region $\boldsymbol{R}$ of the phase plane. What is the minimum number of critical points needed for this to happen?

THE END.