Problem Set Number 02, 18.385j/2.036j
MIT (Fall 2020)

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September 24, 2020

Due Friday October 2, 2020.
Turn it in via the canvas course website.

Contents

1 Irreversible switch using saddle node and transcritical bifurcations

Statement: Irreversible switch using saddle node and transcritical bifurcations

Imagine a system[1] with a controlling parameter r, and with (at most) two distinct stable equilibrium states: $x_1 = x_1(r)$ and $x_2 = x_2(r)$. In particular, such that *infinity is unstable* — that is: for every solution $x = x(t)$ there exists a constant $M > 0$ such that $|x| < M$ for t large enough. Furthermore:

A. There is a value $r = r_s =$ **switch value** such that: for $r > r_s$ both states exist and are stable — so that the system can be in either one of them.

B. For $r < r_s$ only the state x_1 exists and it is stable.

[1] A "switch".

C. Both $x_1(r)$ and $x_2(r)$ are continuous functions of r (though, maybe, not smooth), and $|x_1(r) - x_2(r)|$ is bounded away from zero.

Such a system, if started in the state x_2 for $r > r_s$, remains in x_2 for as long as r varies (slowly enough) in the range $r > r_s$. Once r crosses below the threshold r_s, the system switches to x_1, and remains there for all values of r. A switch back to x_2 is not produced by slow variations in r. The condition in item **C** is important, for otherwise small perturbations could produce an "accidental" switch if x_1 and x_2 get very close.

Remark 1.1 *A "standard" (reversible) switch [e.g.: a thermostat], operates using hysteresis. For such systems there are two switching values $r_1 < r_2$, with only x_2 stable for $r > r_2$, only x_1 stable for $r < r_1$, and both states stable for $r_1 \le r \le r_2$. Then the system jumps from x_2 to x_1 as r is lowered below r_1, and goes back to x_2 as r is raised above r_2.* ♣

Construct an irreversible switch, using a 1-D system of the form

$$\frac{dx}{dt} = f(x,\, r), \tag{1.1}$$

with the behavior caused by two bifurcations: a trans-critical and a saddle node *(no other bifurcations should occur!)* **Then draw the bifurcation diagram.**

Hint: It is very easy to construct an explicit example in which f in equation (1.1) is a cubic polynomial in x, and it is linear in the parameter r.

Remark 1.2 (Switch uniqueness). *Even for a 1-D system such as the one in (1.1), there is an infinite number of possible bifurcation diagrams that yield a switch, with various types of bifurcations involved.*[2] *However, if the restriction that there should be only two bifurcations (one saddle-node and one transcritical) is imposed, then there are only two possible topologies for the switch bifurcation diagram. This problem asks you to produce an example of one such switch.* ♣

2 Perturbed pitchfork, with root preserved (bifurcation diagram)

Statement: Perturbed pitchfork, with root preserved (bifurcation diagram)

Consider the structural stability for a (soft) pitchfork bifurcation, with the restriction that the "main" solution branch is preserved across the bifurcation. Specifically, consider the situation where:

$$\frac{dx}{dt} = g(x,\, r) \quad (g \text{ odd in } x), \tag{2.1}$$

has a (soft) pitchfork bifurcation at $(x,\, r) = (0,\, 0)$. Assume that the problem depends on a hidden parameter h — i.e. let $\boldsymbol{g(x,\, r) = f(x,\, r,\, h)\big|_{h=0}}$, where you only know that h is small (but it may not be zero). Assume also that you know that $\boldsymbol{f(0,\, r,\, h) = 0}$, though f may not be odd for $h \neq 0$. Provided that f is reasonably smooth, and f is generic, it can be shown that the canonical equation[3] describing this situation is

$$\frac{dx}{dt} = r\,x + h\,x^2 - x^3. \tag{2.2}$$

[2] This is the subject of another problem: *Irreversible switches; classification.*

[3] That is, near the bifurcation, the full problem can be mapped into equation (2.2).

Tasks: Assume $h \neq 0$ small (say, $h = 0.05$), and **draw the bifurcation diagram** for (2.2), including the flow lines — recall that the bifurcation diagram is, basically, all the phase portraits (one for each r) stacked in one single 2-D plot. *What happens to the pitchfork?* Furthermore: *estimate the level of noise (in x) under which the distinction between the pitchfork and the new behavior will be hidden* — do this in terms of h.

3 Perturbed pitchfork, with root preserved (canonical form)

Statement: Perturbed pitchfork, with root preserved (canonical form)

Consider the structural stability for a (soft) pitchfork bifurcation, with the restriction that the "main" solution branch is preserved across the bifurcation. Specifically, consider the situation where:

$$\dot{y} = g(y, \lambda) \quad (g \text{ odd in } y), \tag{3.1}$$

has a (soft) pitchfork bifurcation at $(y, \lambda) = (0, 0)$. Assume that the problem depends on a hidden parameter ρ — i.e. let $\boldsymbol{g(y, \lambda) = f(y, \lambda, \rho)\big|_{\rho=0}}$, where you know that ρ is small (but it may not be zero). Assume also that you know that $\boldsymbol{f(0, \lambda, \rho) = 0}$, though f may not be odd for $\rho \neq 0$. Provided that f is reasonably smooth, and f is generic, it can be shown that the canonical equation[4] describing this situation is

$$\dot{x} = r\,x + h\,x^2 - x^3. \tag{3.2}$$

SHOW THIS by using a Taylor expansion in the regime where (y, λ, ρ) are all small. ♣

Hint. The easiest approach is to expand f in powers of y, with coefficients that are functions of λ and ρ. Then use what you know of f to estimate the size of the coefficients (when λ and ρ are small), and then neglect any term that is majored by another term.† Then, upon re-scaling,‡ the resulting equation will be (3.2).

† A term neglected must be smaller than the terms retained for **all** (y, λ, ρ) in some neighborhood of $(0, 0, 0)$. Making an expansion as suggested (as opposed to expanding in all three (y, λ, ρ)), simplifies this step quite a bit.

‡ Note that r and h are generally not directly λ and ρ, but functions of λ and ρ.

4 Toy model for shell buckling

Statement: Toy model for shell buckling

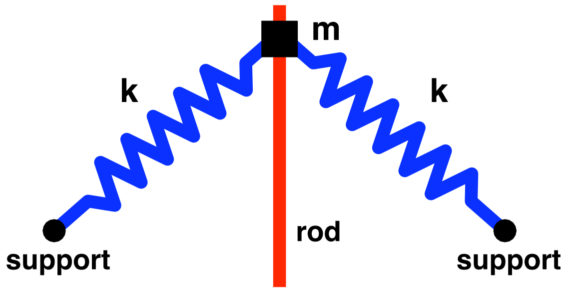
Hold a ping-pong ball between your thumb and index fingers and squeeze it. If you do not apply enough force, the ball will deform slightly with a purely elastic response. But, if you push hard enough, the ball will buckle and you will make a (permanent) dent on it — and the ball will be ruined. This is the phenomena of (thin) shell buckling.

Shell buckling is a very rich phenomena,[5] way beyond the scope of this course. Here we will study an extremely simplified (1-D) version of this phenomena (the emphasis here being on "toy" model) where all the geometrical richness of the original setting is gone, and only the buckling bifurcation remains.

A sketch depicting the model is shown in figure 4.1. Further assumptions and notation are:

[4] That is, near the bifurcation, the full problem can be mapped into equation (3.2).

[5] Lots of interesting and important questions arise. For example: What is the shape of the dent that forms? The dent's edges have sharp corners: why these corners form, and how do they propagate as further pressure is applied?



A bead of mass m (black square) can slide along a rigid vertical rod (in red). The bead is connected by two equal springs (in blue), with spring constant k, to two supports placed symmetrically on each side of the rod. See the text for further details.

Figure 4.1: Toy model for shell buckling.

1. Idealize the bead as a point mass.

2. Let x be the vertical distance, along the rod, of the bead from the horizontal line joining the spring supports. Let $x > 0$ if the bead is above the supports and $x < 0$ if below.

3. Let $\boldsymbol{h > 0}$ be the distance of the spring supports from the rod, and let $\boldsymbol{L > 0}$ be the springs equilibrium length. **Assume $\boldsymbol{L > h}$**, so that the springs are under compression for $x = 0$.

4. Hook's law applies to the springs. Thus they exert a force of magnitude $F = k\,(\ell - L)$, where ℓ is the spring length, along the spring axis, pushing if $\ell < L$, and pulling if $\ell > L$.

5. When the bead slides along the rod, the motion is opposed by a friction force of magnitude $b\,\dot{x}$, where $b > 0$ is a constant.

6. Because the rod is rigid, we need to consider only the vertical components of the various forces that act on the bead. These forces are: (i) Gravity, of magnitude $m\,g$, pointing down. (ii) The forces by the springs. (iii) Friction along the rod. *Note that here we assume that the force gravity is significant, so that there is no up-down symmetry in this problem.*

PROBLEM TASKS:

A. *Derive an ode for the bead position, and write it in appropriate a-dimensional variables.*[6]

B. *Assume that friction is large, so that inertia can be neglected. Exactly which a-dimensional number has to be small for friction to be "large"?*

C. *Analyze the bifurcations that occur for the equation resulting from item B, as the bead mass changes — in this toy model, increasing the bead mass plays the role of squeezing harder on the ping-pong ball. What type of bifurcation(s) occur?*

Hint: It is a bad idea to try to do this by attempting to solve for the critical points and bifurcation thresholds analytically. A qualitative, graphical, analysis is the best way to go.

D. *The picture in figure* 4.1 *corresponds, in this toy model, to the ping-pong ball in a more-or-less spherical shape. What is the "buckled" state?*

E. *What a-dimensional parameter controls when bifurcations happen?* This under the assumption:

$$\textbf{The ratio } \gamma = L/h > 1 \textbf{ is kept fixed.} \tag{4.1}$$

Thus γ is **not** the bifurcation parameter to use; something else is.

[6] Suggestion: to a-dimensionalize use h for length and $b/(2\,k)$ for time.

5 Problem 03.02.06 - Strogatz (Eliminate the cubic term)

Statement for problem 03.02.06

Consider the system

$$\frac{dX}{dt} = RX - X^2 + aX^3 + O(X^4), \tag{5.2}$$

where $R \neq 0$. We want to find a new variable x such that the system transforms into

$$\frac{dx}{dt} = Rx - x^2 + O(x^4). \tag{5.3}$$

This would be a big improvement, since the cubic term has been eliminated and the error term has been bumped to fourth order.[7] In fact, the procedure to do this (sketched below) can be generalized to higher orders.[8] This generalization is the subject matter of problem 03.02.07.
Let $\boldsymbol{x = X + bX^3 + O(X^4)}$, where b is chosen later to eliminate the cubic term in the differential equation for x. This is called a **near-identity transformation**, since x and X are practically equal: they differ by a cubic term.[9] Now we need to rewrite the system in terms of x; this calculation requires a few steps.

1. Show that the near-identity transformation can be inverted to yield $\boldsymbol{X = x + cx^3 + O(x^4)}$, and solve for c.
2. Write $\boldsymbol{\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)}$, and substitute for X and $\dot{X}$ on the right hand side, so that everything depends only on x. Multiply the resulting series expansions and collect terms, to obtain $\boldsymbol{\dot{x} = Rx - x^2 + kx^3 + O(x^4)}$, where k depends on a, b, and R.
3. Now the moment of triumph: choose b so that $k = 0$.
4. Is it really necessary to make the assumption that $R \neq 0$? Explain.

6 Stability index for flows in the circle

Statement: Stability index for flows in the circle

Show that *the stability index $\mathcal{S}$ for any flow in the circle vanishes.* To be precise, consider an equation of the form

$$\frac{d\theta}{dt} = f(\theta), \tag{6.1}$$

where θ is an angle (in radians), and f is periodic of period 2π and Lipschitz continuous. Assume also that the equation has a finite number of critical points:[10] $\theta_1 < \theta_2 < \cdots < \theta_N < \theta_1 + 2\pi$. Now assign a weight $w = 1$ to each stable critical point, a weight $w = -1$ to each unstable critical point, and a weight $w = 0$ to each semi-stable critical point. Then show that

$$\mathcal{S} = \sum_{n=1}^{N} w_n = 0. \tag{6.2}$$

[7] Obviously we are considering here a situation where X (and x) is small.
[8] That is, one can successively eliminate all the higher order terms: $O(x^3)$, $O(x^4)$, ..., etc.
[9] We have skipped the quadratic term X^2, because it is not needed — you should check this later.
[10] The critical points are the zeros of f.

Hint 6.1 *Consider the intervals I_n, $1 \leq n \leq N$, where I_n is the interval $\theta_n < \theta < \theta_{n+1}$ — here $\theta_{N+1} = \theta_1 + 2\pi$, which is the same point as θ_1 because we are in the circle. Then in each such interval either*[11] $f > 0$ *or* $f < 0$. **Define** $\sigma_n = 1$ *if* $f > 0$ *in* I_n, *and* $\sigma_n = -1$ *if* $f < 0$ *in* I_n. *Then relate the* w_n *to the* σ_n *to show (6.2).* What information do the σ_n capture? ♣

7 Bifurcations in the circle problem #06

Statement: Bifurcations in the circle problem #06

For equation (7.1) find the values of r at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram for the fixed points versus r, including the flow direction and the stability of the various branches of solutions (solid lines for stable branches and dashed ones for unstable ones).

$$\frac{d\theta}{dt} = (r + \sin(2\,\theta))\,\sin(\theta), \tag{7.1}$$

where θ is an angle (in radians). Note that *the bifurcation diagram — which is periodic in θ — should be for a 2π range in θ, and a range of r that includes all the bifurcations.*

8 Bifurcations in the torus #01

Statement: Bifurcations in the torus #01

Bifurcations in the torus, phase-locking, and oscillator death. This problem is based on a paper on systems of neural oscillators by G. B. Ermentrout and N. Kopell,[12] where they illustrate the notion of *oscillator death* (see § 10) with the following model

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1\,\cos\theta_2 \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + \sin\theta_2\,\cos\theta_1, \tag{8.1}$$

where $\boldsymbol{\omega_1,\ \omega_2 > 0}$. *Here θ_1 and θ_2 are to be interpreted as the phases of two coupled stable and attracting limit cycle oscillators, which are assumed to "survive" the coupling, so that the notion of their "individual phases" remains —* see § 10.

a. Classify all the different behaviors that the solutions to (8.1) have, as the parameters vary in the positive quadrant of the $[\omega_1,\,\omega_2]$–plane. **Do a diagram** in this quadrant, indicating the regions that correspond to each behavior.

The final answer should look something like this: *(i) In such and such region the solutions are attracted to a limit cycle [Note that this is phase locking]. (ii) In such and such region the solutions are attracted to a stable node [Note that this is oscillator death]. (iii) In such and such region the solutions are quasi-periodic with two periods [Phase locking fails]. (iv) ...*
Plus a drawing of the regions ... will all the statements properly justified.

[11] If f were to switch sign in I_n, then (since it is continuous) it would have a zero in I_n. This zero would no be one of the θ_n, which are supposed to be all the zeros.
[12] *Oscillator death in systems of coupled neural oscillators.* SIAM J. Appl. Math. **50**:125 (1990).

b. Draw the bifurcation curves in the $[\omega_1, \omega_2]$–plane. **Describe** each bifurcation.

Hints. *I did not find an elegant way to analyze the system geometrically. The hints below lead you to an approach that is (mostly) analytical, but allows a systematic and thorough investigation.*

h1. Consider the equations satisfied by $\phi = \theta_1 + \theta_2$ and $\psi = \theta_1 - \theta_2$.

h2. You may find the following result useful
Let $\boldsymbol{\alpha > 1}$. Then the solutions to the equation $\dot{\boldsymbol{\chi}} = \boldsymbol{\alpha + \sin\chi}$
can be written in the form $\boldsymbol{\chi = \mu\,(t - t_0) + X(\mu\,(t - t_0))}$,
where $\boldsymbol{\mu > 0}$ is a constant, $\boldsymbol{X}$ is $2\,\pi$-periodic,
$\boldsymbol{X(0) = 0}$, and t_0 is an arbitrary constant.

Furthermore: $\boldsymbol{\mu}$ is an increasing function of $\boldsymbol{\alpha}$, with $\displaystyle\lim_{\boldsymbol{\alpha\to 1}} \boldsymbol{\mu = 0}$ and $\displaystyle\lim_{\boldsymbol{\alpha\to\infty}} \boldsymbol{\mu = \infty}$.

All this follows from § 11, upon using a change of variables to transform $\dot{\chi} = \alpha + \sin\chi$ into equation (11.1). In particular, note that the "μ" in § 11 (call it $\tilde{\mu}$) is related to the one here by $\mu = \alpha\,\tilde{\mu}$, with $\kappa = \frac{1}{\alpha}$.

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9 First order equation with a periodic right hand side

Statement: First order equation with a periodic right hand side

Consider the equation

$$\boldsymbol{\dot{\phi} = 1 - \kappa\,\sin\phi,} \quad \text{where } \boldsymbol{0 < \kappa < 1.} \tag{9.1}$$

Since $\dot{\phi} \geq 1 - \kappa > 0$, ϕ *is monotone increasing.* **Prove the statements below.**

1. There is a constant $0 < \mu < 1$, and a function $\Phi = \Phi(\zeta)$ — periodic of period $2\,\pi$ — such that any solution to (9.1) has the form

$$\boldsymbol{\phi = \mu\,(t - t_0) + \Phi(\mu\,(t - t_0)),} \tag{9.2}$$

   where t_0 is a constant and $\boldsymbol{\Phi(0) = 0}$. It follows that $\boldsymbol{\sin(\phi)}$ **is periodic in $\boldsymbol{t}$, with period $\boldsymbol{T = \frac{2\,\pi}{\mu}}$**

2. The period-average M for $\sin(\phi)$ is given by $M = M(\kappa)$ only (μ depends only on κ).

$$\boldsymbol{M =} \textbf{ average}\boldsymbol{(\sin\phi) = \frac{1-\mu}{\kappa} > 0.} \tag{9.3}$$

3. Let $\boldsymbol{\phi_*}$ **be the solution to (9.1) defined by $\boldsymbol{\phi_*(0) = 0}$** — i.e.: set $t_0 = 0$ in (9.2). Then

$$\Theta(\mu\,t) = \int_0^t (\sin(\phi_*(s)) - M)\,ds = -\frac{1}{\kappa}\,\Phi(\mu\,t), \tag{9.4}$$

   where Θ is defined by the first equality.

4. Assume that $0 < \kappa \ll 1$. Then a Poincaré-Lindstedt expansion yields

$$\phi_* = \mu\,t - \kappa\,(1 - \cos(\mu\,t)) + O(\kappa^2) \quad \text{and} \quad \mu = 1 - \frac{1}{2}\,\kappa^2 + O(\kappa^4). \tag{9.5}$$

   It follows that $T = 2\,\pi + \pi\,\kappa^2 + O(\kappa^4)$ and $M = \dfrac{1}{2}\,\kappa + O(\kappa^3)$.

5. Assume that $0 < 1 - \kappa \ll 1$. Then

$$\boldsymbol{\mu = O(\sqrt{1 - \kappa})} \tag{9.6}$$

Hints.

a. Define $T > 0$ as the (unique) time at which $\phi_*(T) = 2\pi$ — why is the solution unique?

b. Show that $\phi_*(t+T) = 2\pi + \phi_*(t)$ — sub-hint: both sides are solutions!

c. Define Φ by $\Phi(\mu t) = \phi_*(t) - \mu t$, with $\mu = 2\pi/T$, and show that Φ is periodic of period 2π.

d. Write the general solution in terms of ϕ_*.

e. Show that $T = O\big(1/\sqrt{1-\kappa}\big)$ as $\kappa \to 1$ — sub-hint: critical slowing-down.

f. To show that $\mu < 1$, use (9.1) and separation of variables to write T as an integral over ϕ from 0 to 2π. Then show $T > 2\pi$

g. To show (9.3), take the average of (9.1).

h. To obtain the second equality in (9.4), substitute $\phi_* = \mu t + \Phi(\mu t)$ into (9.1), and obtain a formula for $\sin(\phi_*)$ in terms of Φ.

Part I

Supplementary notes on oscillators and ode

10 Notes: coupled oscillators, phase locking, oscillator death, etc.

10.1 On phases and frequencies

Consider a system made by two coupled oscillators, where each of the oscillators (when not coupled) has a stable attracting limit cycle. Let the limit cycle solutions for the two oscillators be given by $\boldsymbol{\vec{x}_1 = \vec{F}_1(\omega_1\, t)}$ and $\boldsymbol{\vec{x}_2 = \vec{F}_2(\omega_2\, t)}$, where $\boldsymbol{\vec{x}_1}$ and $\boldsymbol{\vec{x}_2}$ are the vectors of variables for each of the two systems, the $\boldsymbol{\vec{F}_j}$ *are periodic functions of period* 2π, and the ω_j are constants (related to the limit cycle periods by $\omega_j = 2\pi/T_j$). In the un-coupled system, the two limit cycle orbits make up a *stable attracting invariant torus* for the evolution. Assume now that either the coupling is weak, or that the two limit cycles are strongly stable. *Then the stable attracting invariant torus survives for the coupled system.*[13] The solutions (on this torus) can be (approximately) represented by

$$\vec{x}_1 \approx \vec{F}_1(\theta_1) \quad \text{and} \quad \vec{x}_2 \approx \vec{F}_2(\theta_2), \tag{10.1}$$

where $\boldsymbol{\theta_1 = \theta_1(t)}$ and $\boldsymbol{\theta_2 = \theta_2(t)}$ satisfy some equations, of the general form

$$\dot{\theta}_1 = \omega_1 + K_1(\theta_1,\, \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + K_2(\theta_1,\, \theta_2). \tag{10.2}$$

Here K_1 and K_2 are the “projections” of the coupling terms along the oscillator limit cycles. For example, take $K_1(\theta_1,\, \theta_2) = \sin\theta_1\, \cos\theta_2$ and $K_2(\theta_1,\, \theta_2) = \sin\theta_2\, \cos\theta_1$. Another example is the one in § 8.6 of Strogatz’ book (*Nonlinear Dynamics and Chaos*), where a model system with

$$K_1(\theta_1,\, \theta_2) = -\kappa_1\, \sin(\theta_1 - \theta_2) \text{ and } K_2(\theta_1,\, \theta_2) = \kappa_2\, \sin(\theta_1 - \theta_2)$$

is introduced, with constants $\kappa_1,\, \kappa_2 > 0$. Note that:

1. In (10.2), K_1 and K_2 must be 2π-periodic functions of θ_1 and θ_2.

[13] With a (slightly) changed shape and position.

2. The **phase space for (10.2) is the invariant torus $\mathcal{T}$**, on which θ_1 and θ_2 are the angles. We can also think of $\mathcal{T}$ as a $2\pi \times 2\pi$ square with its opposite sides identified. On $\mathcal{T}$ a **solution is periodic** if and only if $\theta_1(t+T) = \theta_1(t) + 2n\pi$ and $\theta_2(t+T) = \theta_2(t) + 2m\pi$, where $T > 0$ is the period, and both n and m are integers.
3. In the "Coupled oscillators # 01" problem an example of the process leading to (10.2) is presented.
4. The $\boldsymbol{\theta_j}$'s are the **oscillator phases.** One can also define oscillator frequencies, even when the θ_j's do not have the form $\theta_j = \omega_j\, t$, with ω_j constant.
   The idea is that, near any time t_0 we can write $\theta_j = \theta_j(t_0) + \dot{\theta}_j(t_0)\,(t - t_0) + \ldots,$ identifying $\dot{\theta}_j(t_0)$ as the local frequency. Hence, we define the **oscillator frequencies by $\tilde{\omega}_j = \dot{\theta}_j$.** These frequencies are, of course, **generally not constants.**
5. The notion of phases can survive **even if** the limit cycles cease to exist (i.e.: oscillator death). For example: if the equations for θ_1 and θ_2 have an attracting critical point. We will see examples where this happens in the problems, e.g.: "Bifurcations in the torus # 01".

10.2 Phase locking and oscillator death

The coupling of two oscillators, each with a stable attracting limit cycle, can produce many behaviors. Two of particular interest are

1. Often, if the frequencies are close enough, the system **phase locks.** This means that a stable periodic solution arises, in which both oscillators run at some composite frequency, with their phase difference kept constant. The composite frequency need not be constant. In fact, it may periodically oscillate about a constant average value.
2. However, the coupling may also suppress the oscillations, with the resulting system having a stable steady state. This even if none of the component oscillators has a stable steady state. This is **oscillator death.** It can happen not only for coupled pairs of oscillators, but also for chains of oscillators with coupling to the nearest neighbors.

On the other hand, we note that it is also possible to produce an oscillating system, with a stable oscillation, by coupling non-oscillating systems (e.g., the coupling of excitable systems can do this).

11 Notes: first order equation with a periodic right hand side

You will be asked to justify the statements below in one of the problems. They are stated here because these results are used/needed in the answers to some of the other problems.

Consider the equation

$$\dot{\phi} = 1 - \kappa\,\sin\phi, \quad \text{where } 0 < \kappa < 1. \tag{11.1}$$

Since $\dot{\phi} \geq 1 - \kappa > 0$, ϕ *is monotone increasing.* **It can be shown that:**

1. There is a constant $0 < \mu < 1$, and a function $\Phi = \Phi(\zeta)$ — periodic of period 2π — such that any solution to (11.1) has the form
$$\phi = \mu\,(t - t_0) + \Phi(\mu\,(t - t_0)), \tag{11.2}$$
   where t_0 is a constant and $\Phi(0) = 0$. It follows that $\sin(\phi)$ **is periodic in t, with period $T = \frac{2\pi}{\mu}$**

2. The period-average M for $\sin(\phi)$ is given by $M = M(\kappa)$ only (μ depends only on κ).

$$\boldsymbol{M = \textsf{average}(\sin\phi) = \tfrac{1-\mu}{\kappa} > 0}. \tag{11.3}$$

3. Let $\boldsymbol{\phi_*}$ **be the solution to (11.1) defined by** $\boldsymbol{\phi_*(0) = 0}$ — i.e.: set $t_0 = 0$ in (11.2). Then

$$\Theta(\mu\, t) = \int_0^t (\sin(\phi_*(s)) - M)\, ds = -\frac{1}{\kappa}\,\Phi(\mu\, t), \tag{11.4}$$

where Θ is defined by the first equality.

4. Assume that $0 < \kappa \ll 1$. Then a Poincaré-Lindstedt expansion yields

$$\phi_* = \mu\, t - \kappa\,(1 - \cos(\mu\, t)) + O(\kappa^2) \quad \text{and} \quad \mu = 1 - \frac{1}{2}\,\kappa^2 + O(\kappa^4). \tag{11.5}$$

It follows that $T = 2\,\pi + \pi\,\kappa^2 + O(\kappa^4)$ and $M = \dfrac{1}{2}\,\kappa + O(\kappa^3)$.

5. Assume that $0 < 1 - \kappa \ll 1$. Then

$$\boldsymbol{\mu = O(\sqrt{1-\kappa})} \tag{11.6}$$

THE END.