Problem Set Number 01, 18.385j/2.036j
MIT (Fall 2020)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

September 9, 2020

Due September 18, 2020.
Turn it in via the canvas course website.

Contents

1 Get equation from phase line portrait problem #03 **1**

2 Get equation from phase line portrait problem #06 **2**

3 Ill posed initial value problem #01 **2**

4 Implicit function problem #01 **2**
Compute the coefficients of the Taylor expansion for the implicit function 2

5 Inverse function problem #01 **3**
Compute the coefficients of the Taylor expansion for the inverse function 3

6 Linear model for a hanging rope under tension **3**

7 Problem 02.02.10 - Strogatz (Fixed points) **3**

8 Problem 02.05.06 - Strogatz (The leaky bucket) **4**

9 Problem 02.06.01 - Strogatz (1D oscillator "paradox") **5**

10 Well posed initial value problem #01 **5**

1 Get equation from phase line portrait problem #03

Statement: Get equation from phase line portrait problem #03

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \tag{1.1}$$

where f is some function which is (at least) Lipschitz continuous. Assume that (1.1) has exactly two critical points (i.e.: x_1 and x_2, with $-\infty < x_1 < x_2 < \infty$). Assume also that x_1 is stable and that x_2 is semi-stable.[1] **Is this possible? Does a function $f = f(x)$ yielding this exist?**
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

[1] A critical point is semi-stable if the solutions diverge from the critical point on one side, and converge on the other.

2 Get equation from phase line portrait problem #06

Statement: Get equation from phase line portrait problem #06

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \tag{2.1}$$

where f is some function which is (at least) Lipschitz continuous. Assume that (2.1) has exactly two critical points (i.e.: x_1 and x_2, with $-\infty < x_1 < x_2 < \infty$). Assume also that both critical points are unstable.[2] **Is this possible? Does a function $f = f(x)$ yielding this exist?**

If the answer is no, prove it.

If the answer is yes, prove it by giving an example.

---

3 Ill posed initial value problem #01

Statement: Ill posed initial value problem #01

Consider the initial value problem

$$\frac{dx}{dt} = 2\sqrt{|x|}, \quad \text{for } 0 < t, \quad \text{with } x(0) = 0. \tag{3.1}$$

Show that this problem has infinitely many solutions, which can be parameterized by the time $0 \leq \tau \leq \infty$ beyond which the solution ceases to vanish.

Hint: use separation of variables to find all the solutions that are possible when $x \neq 0$.

---

4 Implicit function problem #01

Statement: Implicit function problem #01

Consider the following equation

$$\boldsymbol{f(x,\,\lambda) = x + \lambda\,\cos(x) = 0}, \tag{4.1}$$

with the particular solution $\boldsymbol{(x,\,\lambda) = (0,\,0)}$. Since $f_x(0,\,0) = 1$, the implicit function theorem guarantees that: *there is a neighborhood of $\lambda = 0$ where (4.1) has a unique solution, $x = X(\lambda)$, such that $X(0) = 0$.* Furthermore, since f is an analytic function, X is an analytic function of λ. This means that X has a Taylor series

$$\boldsymbol{X = \sum_{n=0}^{\infty} x_n\,\lambda^n}, \tag{4.2}$$

which converges for $|\lambda|$ small enough. **Find $\boldsymbol{x_1}$, $\boldsymbol{x_3}$, $\boldsymbol{x_5}$, and $\boldsymbol{x_n}$ for all even $\boldsymbol{n}$.**

---

[2] A critical point is unstable if the solutions diverge from the critical point on both sides.

5 Inverse function problem #01

Statement: Inverse function problem #01

Consider the following equation
$$\boldsymbol{y = x + \sin(x) = f(x)},\tag{5.1}$$
where $\boldsymbol{f(0) = 0}$ and $\boldsymbol{f'(0) = 2 \neq 0}$. The inverse function theorem guarantees that *there is a neighborhood of* $x = 0$ *where* f *has a unique inverse,* $x = X(y)$*, such that* $X(0) = 0$*.* Furthermore, since f is an analytic function, X is an analytic function. This

means that X has a Taylor series
$$\boldsymbol{X = \sum_{n=0}^{\infty} x_n\, y^n},\tag{5.2}$$
which converges for $|\lambda|$ small enough. **Find $\boldsymbol{x_1}$, $\boldsymbol{x_3}$, $\boldsymbol{x_5}$, and $\boldsymbol{x_n}$ for all even $\boldsymbol{n}$.**

6 Linear model for a hanging rope under tension

Statement: Linear model for a hanging rope under tension

Consider a thin rope stretched (at equilibrium) between two nails at the same height, with the nails separated by some horizontal distance. Idealize the rope as a curve in space, and describe its shape by the vertical deviation of the rope $u = u(x)$ from the horizontal straight line connecting the nails. *Assume that the deviation of the shape from a straight line is small, so that at any point along the rope* $\boldsymbol{\sin(\theta) \approx \tan(\theta) = \frac{du}{dx}}$ *is a good approximation* — here θ is the angle with the horizontal made by the tangent line to the rope.

Let the distance between nails be $\boldsymbol{L}$ (the left nail is at $x = 0$ and the right one at $x = L$), let $\boldsymbol{T > 0}$ be the tension along the rope,[3] and let $\boldsymbol{g}$ be the acceleration of gravity.[4] Finally assume that the rope is homogeneous, with density ρ (mass per unit length).

Write a mathematical model for $\boldsymbol{u}$, and show that it is well posed (the solution is unique, and it depends continuously on all the parameters involved).

Hint #1: The model is a boundary value problem in $0 < x < L$*, for a second order, non-homogeneous, linear ode.*

Hint #2: To derive the ode for u*, consider the balance of the vertical forces*[5] *on an arbitrary segment of the rope* $a \leq x \leq b$*. The forces are: (i) gravity, (ii) the tension force by the piece of rope on* $x > b$*, and (iii) the tension force by the piece of rope on* $x > a$*. Then divide by* $(b - a)$ *the resulting equation, and take the limit* $(b - a) \to 0$ *— with both* b, $a \to x_0 =$ *some arbitrary point along the rope.*

Hint #3: What the boundary conditions on u *are is (or should) be obvious.*

7 Problem 02.02.10 - Strogatz (Fixed points)

Statement for problem 02.02.10

(Fixed points). For each of (A)–(E) below, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not. In all cases assume that $f(x)$ is a smooth function.

[3] The tension is the magnitude of the force with which, at any point, one side of the rope pulls on the other. The force vector is directed along the rope's tangent, towards the pulling side, and has length T.

[4] Hence, on any mass m, there is a downwards force of magnitude $m\,g$.

[5] Because the rope is at equilibrium, the forces must add to zero.

A. Every real number is a fixed point.

B. Every integer is a fixed point, and there are no others.

C. There are precisely three fixed points, and all of them are stable.

D. There are no fixed points.

E. There are precisely 100 fixed points.

8 Problem 02.05.06 - Strogatz (The leaky bucket)

Statement for problem 02.05.06

(The leaky bucket). The following example[6] shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see a water bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying[7] a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Here is a crude model for the situation. Let $h(t)$ = height of the water remaining in the bucket at time t; a = area of the hole; A = cross-sectional area of the bucket (assumed constant); $v(t)$ = velocity of the water passing through the hole.

a. Show that $a\,v(t) = A\,\dot{h}$. What physical law are you invoking? **Warning:** since $\dot{h} < 0$, this presumes that *we assign a negative value to the velocity* v. This is a weird choice, implicit in the problem statement, but acceptable.

b. To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount Δh, and that the water has density ρ. Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation $v^2 = 2\,g\,h$ — g = gravity acceleration.

c. Combining **a** and **b**, show that $\dot{h} = -C\sqrt{h}$, where $C = \dfrac{a}{A}\sqrt{2\,g}$.

d. Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is non-unique *in backwards* time, i.e., for $t < 0$.

The description/derivation above ignores surface tension. Briefly discuss the effect surface tension has on the outcome.

[6] Hubbard, J. H., and West, B. H. (1991) *Differential Equations: A Dynamical Systems Approach, Part I* (Springer, New York).

[7] Note that, in this problem, evaporation effects are neglected.

9 Problem 02.06.01 - Strogatz (1D oscillator "paradox")

Statement for problem 02.06.01

Explain this paradox: a simple harmonic oscillator $m\ddot{x} = -kx$ is a system that oscillates in one dimension (along the x-axis). But the text says that one-dimensional systems cannot oscillate.

10 Well posed initial value problem #01

Statement: Well posed initial value problem #01

Consider the initial value problem

$$\frac{dx}{dt} = -2\,\mathrm{sign}(x)\,\sqrt{|x|}, \quad \text{for } 0 < t, \quad \text{with } x(0) = x_0 = \text{arbitrary constant.} \tag{10.1}$$

Find ALL the solutions to this problem — note that the solutions are defined for **all** times $\boldsymbol{t > 0}$.

Hint: use separation of variables to find all the solutions that are possible when $x \neq 0$.

Notice that the right hand side of the ode above is not Lipschitz continuous at $x = 0$. Nevertheless, this problem is well posed. **Show this, using the solutions that you obtained in the first part of the problem.**

THE END.