Answers to P-Set # 06, 18.385j/2.036j
MIT (Fall 2020)

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1 Bifurcations of a Critical Point for a 1-D map

1.1 Statement: Bifurcations of a Critical Point for a 1D map

When studying bifurcations of critical points in the lectures, we argued that we could understand most of the relevant possibilities simply by looking at 1-D flows. The argument was based on the idea that, generally, a stable critical point will become unstable in only one direction at a bifurcation — so that the flow will be trivial in all the other directions, and we need to concentrate only on what occurs in the unstable direction. The only important situation that is missed by this argument, is the case when two complex conjugate eigenvalues become unstable (Hopf bifurcation). Because in real valued systems eigenvalues arise either in complex conjugate pairs or as single real ones, the cases where only one (or a complex pair) go unstable are the the only ones likely to occur (barring situations with special symmetries that "lock" eigenvalues into synchronous behavior).
The same argument can be made when studying bifurcations of limit cycles in any number of dimensions. In this case one considers the Poincaré map near the limit cycle,[1] with the role of the eigenvalues taken over by the Floquet multipliers. Again, we argue that we can understand a good deal of what happens by replacing the (multi-dimensional) Poincaré map by a one dimensional map with a stable fixed point, and asking what can happen if the fixed point changes stability as some parameter is varied. In this problem you will be asked to do this.

Remark 1.1 *Some important cases are missed by this approach: the case where a pair of complex Floquet multipliers becomes unstable (Hopf bifurcation of a limit cycle), and the cases where a bifurcation occurs because of an interaction of the limit cycle with some other object (e.g., a critical point). Several examples of these situations can be found in section 8.4 of Strogatz' book (Global Bifurcations of Cycles).* ♣

Consider a one dimensional (smooth) map from the real line to itself

$$x \longrightarrow y = f(x, \mu) \tag{1.1}$$

that depends on some (real valued) parameter μ. **Assume that $x = 0$ is a fixed point for all values of μ — that is, $f(0, \mu) \equiv 0$.** Furthermore, assume that $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$. That is:

$$\left|\frac{\partial f}{\partial x}(0,\, \mu)\right| \quad < \quad 1 \quad \text{for} \quad \mu < 0, \quad \text{and} \tag{1.2}$$

$$\left|\frac{\partial f}{\partial x}(0,\, \mu)\right| \quad > \quad 1 \quad \text{for} \quad \mu > 0. \tag{1.3}$$

A further assumption, that involves no loss of generality (since the parameter μ can always be re-defined to make it true) is that

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,\, 0) \neq 0. \tag{1.4}$$

This guarantees that the loss of stability is linear in μ, as μ crosses zero. This is what is called a *transversality condition.* It means this:

> Graph of the Floquet multiplier $\dfrac{\partial f}{\partial x}(0, \mu)$ as a function of μ. Then the resulting curve crosses one of the lines $y = \pm 1$ transversally (curves not tangent at the common point) for $\mu = 0$.

By doing an appropriate expansion of the map f for x and μ small (or by any other means), **show that** (generally[2]) **the following happens:**

a. ***For $\dfrac{\partial f}{\partial x}(0,\, 0) = 1$, either:***

- **a1.** *Transcritical bifurcation (no special symmetries assumed for f):* There exists another fixed point, $x_* = x_*(\mu) = O(\mu)$, such that: $x_* \neq 0$ is unstable for $\mu < 0$ and $x_* \neq 0$ is stable for $\mu > 0$. The two points "collide" at $\mu = 0$ and exchange stability.

[1] The limit cycle is a fixed point for this map.
[2] There are special conditions under which all this fails. You must find them as part of your analysis. What are they?

a2. *Supercritical or soft pitchfork bifurcation, assuming that f is an odd function of x:* Two stable fixed points exist for $\mu > 0$, one on each side of $x = 0$, at a distance $O(\sqrt{\mu})$. All three points merge for $\mu = 0$.

a3. *Subcritical or hard pitchfork bifurcation, assuming that f is an odd function of x:* Two unstable fixed points exist for $\mu < 0$, one on each side of $x = 0$, at a distance $O(\sqrt{-\mu})$. All three points merge for $\mu = 0$.

What does all this mean in the context of the Poincaré map for a limit cycle?

b. ***For $\dfrac{\partial f}{\partial x}(0, 0) = -1$ (no special symmetries assumed for f), either:***

b1. *Supercritical or soft flip bifurcation:* For $\mu > 0$ two points $x_1(\mu) \approx -x_2(\mu) = O(\sqrt{\mu})$ exist, on each side of the fixed point $x = 0$, with $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1,\, x_2\}$ is a period two orbit for the map (1.1). **Show that this orbit is stable.**

b2. *Subcritical or hard flip bifurcation:* For $\mu < 0$ two points $x_1(\mu) \approx -x_2(\mu) = O(\sqrt{-\mu})$ exist, on each side of the fixed point $x = 0$, with $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1,\, x_2\}$ is a period two orbit for the map (1.1). **Show that this orbit is unstable.**

In the context of the Poincaré map for a limit cycle, a flip bifurcation corresponds to a period doubling bifurcation of the limit cycle.

Hint 1.1 *If you expand f in a Taylor expansion near $x = 0$ and $\mu = 0$, up to the leading order beyond the trivial first term ($f \sim \pm x$), and you make sure to keep **all** the relevant terms (and nothing else), and you make sure to identify certain terms that must vanish, the problem reduces to (mostly) trivial high-school algebra. In figuring out what to keep and what not to keep, note that you have two small quantities (x and μ), whose sizes are related. The process is very similar to the Hopf bifurcation expansion calculation (e.g. see course notes), but much simpler computationally.*

For part (b) you will have to keep in the expansion not just the leading order terms beyond the trivial first term, but the next order as well. *Reason: In this case, you will be looking for solutions to the equation $f(f(x, \mu), \mu) = x$. But, when you calculate $f(f(x, \mu), \mu)$, you will see that the second order terms cancel out — thus the need for an extra term in the expansion. This is the same phenomena that forces the Hopf bifurcation calculation to third order.* ♣

IMPORTANT

To standardize the notation, **use the following symbols in your answer:**

$$\boldsymbol{\nu = \frac{\partial f}{\partial x}(0,\, 0)} \quad \text{— note that } \nu = \pm 1,$$

$$\boldsymbol{a = \frac{1}{2}\,\frac{\partial^2 f}{\partial x^2}(0,\, 0),} \qquad \boldsymbol{b = \frac{\partial^2 f}{\partial x\, \partial \mu}(0,\, 0),} \qquad \boldsymbol{c = \frac{1}{6}\,\frac{\partial^3 f}{\partial x^3}(0,\, 0),} \qquad \text{and} \qquad \boldsymbol{d = \frac{1}{2}\,\frac{\partial^3 f}{\partial x^2\, \partial \mu}(0,\, 0).}$$

Then *obtain leading order expressions for the fixed points and flip-bifurcation orbits in terms of these quantities.*

1.2 Answer: Bifurcations of a Critical Point for a 1D map

We begin by writing the first few terms in the Taylor expansion for f near $x = \mu = 0$. We have

$$f = \nu\, x + a\, x^2 + b\, \mu\, x + c\, x^3 + d\, \mu\, x^2 + e\, \mu^2\, x + O(x^4,\, \mu\, x^3,\, \mu^2\, x^2,\, \mu^3\, x), \tag{1.5}$$

where ν, a, b, c, and d, are as defined in the problem statement. Furthermore:

- $\boldsymbol{\nu = \pm 1}$. This follows from (1.2) and (1.3).
- $\boldsymbol{b \neq 0}$, as follows from (1.4).

- The expansion has no terms involving powers of μ alone. This follows from $f(0,\mu) \equiv 0$.

a. ***Case $\nu = 1$. Then:***

$$f = x + a\,x^2 + b\,\mu\,x + c\,x^3 + d\,\mu\,x^2 + e\,\mu^2\,x + O(x^4, \mu\,x^3, \mu^2\,x^2, \mu^3\,x), \tag{1.6}$$

where $\boldsymbol{b > 0}$, as follows from (1.2) and (1.3) — given that $\dfrac{\partial f}{\partial x}(0,\,\mu) = 1 + b\,\mu + O(\mu^2)$.

We have the following **sub-cases:**

a1. ***Generic sub-case: $a \neq 0$.*** Then the equation $x = f(x,\mu)$ has two solutions near $x = \mu = 0$, as follows from (1.6). These are

$$x = 0 \quad \text{and} \quad x = x_*(\mu) = -\frac{b}{a}\,\mu + O(\mu^2).$$

Furthermore, note that

$$\frac{\partial f}{\partial x}(x_*,\,\mu) = 1 + 2\,a\,x_* + b\,\mu + O(\mu^2) = 1 - b\,\mu + O(\mu^2).$$

Thus x_* is an unstable fixed point for $\mu < 0$ and a stable one for $\mu > 0$.
This **sub-case corresponds to a TRANSCRITICAL bifurcation.**
An expansion for x_* in powers of μ (to any order) is easy to obtain by writing

$$x_* = \alpha_1\,\mu + \alpha_2\,\mu^2 + \dots,$$

substituting this into $f(x_*,\,\mu) = x_*$, using the expression in (1.6) for f, and then equating equal powers of μ to obtain the coefficients α_n.

a2. ***Sub-case: f is an odd function of x, and $c < 0$.*** Then the expansion for f in (1.6) reduces to

$$f = x + b\,\mu\,x + c\,x^3 + O(x^5,\,\mu\,x^3,\,\mu^2\,x) \tag{1.7}$$

Clearly, for $\boldsymbol{0 < \mu \ll 1}$, the equation $f(x,\,\mu) = x$ has three roots near $x = 0$. These are

$$x = 0 \quad \text{and} \quad x = x_1(\mu) = -x_2(\mu) = \sqrt{-\frac{b}{c}\,\mu} + O(\mu^{3/2}).$$

Note that

$$\frac{\partial f}{\partial x}(x_n,\,\mu) = 1 + b\,\mu + 3\,c\,x_n^2 + O(\mu^2) = 1 - 2\,b\,\mu + O(\mu^2).$$

Thus both x_1 and x_2 are stable for $0 < \mu \ll 1$.
This **sub-case corresponds to a SUPERCRITICAL (or soft) PITCHFORK bifurcation.**

a3. ***Sub-case: f is an odd function of x, and $c > 0$.*** This sub-case is entirely analogous to the one in item **a2**, except that the two extra solutions occur for $\mu < 0$ and they are unstable — the algebra for showing this is an exact replica of the one in item **a2**.
This **sub-case corresponds to a SUBCRITICAL (or hard) PITCHFORK bifurcation.**

b. ***Case $\nu = -1$. Then:***

$$f = -x + a\,x^2 + b\,\mu\,x + c\,x^3 + O(x^4,\,\mu\,x^2,\,\mu^2\,x), \tag{1.8}$$

where $\boldsymbol{b < 0}$, as follows from (1.2) and (1.3) — given that $\dfrac{\partial f}{\partial x}(0,\,\mu) = -1 + b\,\mu + O(\mu^2)$. It should then be clear that $f(x,\,\mu) = x$ has no roots other than $x = 0$ near $x = \mu = 0$, as follows from the implicit function theorem and the fact that $g = g(x,\, \mu) = f - x$ has a nonzero partial derivative with respect to x at $x = \mu = 0$. Thus consider the second iterate of f, namely $f^2 = f^2(x) = f(f(x))$, and its expansion:

$$f^2 = x - 2b\mu x - 2(a^2 + c)x^3 + O(x^4, \mu x^2, \mu^2 x)\,. \tag{1.9}$$

The leading order terms in this expansion have exactly the same form as the leading order terms in (1.7), except that the constants are different (and f^2 is not necessarily odd). The algebra that follows is thus very similar, except that fixed points of f^2 which are different from $x = 0$ correspond to period two orbits of f.

b1. ***Sub-case: $a^2 + c > 0$.*** Then f^2 has two nonzero fixed points for $\mathbf{0 < \mu \ll 1}$, which correspond to a period two orbit. These are given by

$$x = x_1(\mu) = \sqrt{-\frac{b}{a^2 + c}\,\mu} + O(\mu^{3/2}) \quad \text{and} \quad x = x_2(\mu) = -\sqrt{-\frac{b}{a^2 + c}\,\mu} + O(\mu^{3/2}).$$

Note that, while $x_1 \approx -x_2$, this is (generally) only an approximate equality — with the two roots differing at higher orders. We also note that

$$\frac{\partial f^2}{\partial x}(x_n,\, \mu) = 1 - 2\,b\,\mu - 6\,(a^2 + c)\,x_n^2 + O(\mu^2) = 1 + 4\,b\,\mu + O(\mu^2).$$

Thus the orbit is stable for $\mu > 0$ (recall that $b < 0$).
This **sub-case corresponds to a SUPERCRITICAL (or soft) FLIP bifurcation.**

b2. ***Sub-case: $a^2 + c < 0$.*** This case is entirely analogous to the one in item **b1**, except that the period two orbit occurs for $\mu < 0$ and it is unstable — the algebra for showing this is an exact replica of the one in item **b1**. This **sub-case corresponds to a SUBCRITICAL (or hard) FLIP bifurcation.**

b3. ***Sub-case: $a^2 + c = 0$.*** This is not generic. In the absence of any known symmetry of the equations guaranteeing this, it is not important *(structurally unstable situation)*.

2 Sierpinski gasket

2.1 Statement: Sierpinski gasket

Consider the fractal (a "Sierpinski gasket") in the plane, made in the following **recursive fashion:**

1. Start with an equilateral triangle, with sides of length L.
2. Draw the lines joining the sides mid-points, and divide it into four equal equilateral sub-triangles.
3. Remove the sub-triangle at the center.
4. Repeat the process with each of the other three remaining sub-triangles.

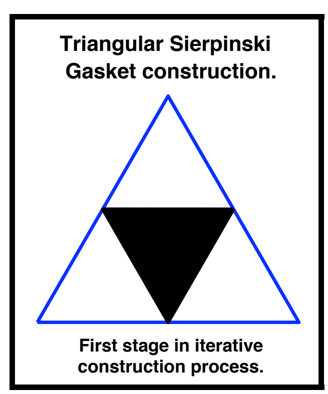

Figure 2.1: The picture on the right illustrates the recursion, showing the result of the first iteration in the process described above.

Now, do the following:

A. Calculate the box dimension of the fractal.

B. Calculate the self-similar dimension of the fractal.

C. Calculate the surface area of the fractal.

D. Show that the fractal has as many points as a full square — **This part is hard(er).**

E. Let d_s be the dimension calculated in part **A**. Modify the construction of the fractal, in such a way that the modified fractal can be selected to have **any** given box dimension $0 < d < d_s$.
Hint: take out bigger chunks at each stage.

F. Construct fractals (subsets of the plane) such that their box dimensions can be selected to have **any** given box dimension $d_s < d < 2$.

2.2 Answer: Sierpinski gasket

We start with the easier questions, and leave part **D** (cardinality of the fractal) to the end.

Part A: ***Box dimension of the fractal***

It is clear that the fractal can be covered with either:
1. One equilateral triangle, whose sides are of length L.
2. Three equilateral triangles, whose sides are of length $L/2$.
. . . .
n. 3^n equilateral triangles, whose sides are of length $L/2^n$.
It follows that the **box dimension** is given by:

$$d_b = \lim_{n\to\infty} \frac{\log(3^n)}{-log(L/2^n)} = \frac{\log(3)}{log(2)} \approx 1.5850. \tag{2.1}$$

Part B: ***Self-similar dimension of the fractal***

The process is similar to the one used for the box dimension. It is clear that, for any natural number n, the fractal is made of by 3^n identical copies of itself, reduced in size by a factor of 2^n. Thus the **self-similar dimension** is given by:

$$d_s = \frac{\log(3)}{log(2)} \approx 1.5850. \tag{2.2}$$

Part C: ***Surface area of the fractal***

Since the fractal is included in the starting triangle T_0 — and in all the objects that result from applying the iteration process that defines the fractal — it follows that:

$$A_\infty \leq A_n \quad \text{for every } n = 0,\, 1,\, 2,\, \ldots \tag{2.3}$$

Here A_∞ denotes the area of the fractal, and A_n is the area of T_n, where T_n denotes the set produced by iterating n times the process that leads to the fractal. However, it should be clear that $A_{n+1} = \frac{1}{4}\,A_n$, so that $A_n \to 0$. Thus **the fractal has no surface area: $\boldsymbol{A_\infty = 0}$**.

Part E: ***Generalizations with a smaller dimension***

Let $\boldsymbol{d_b = d_s = \log(3)/\log(2)}$ be the dimensions calculated in parts **A** and **B**. Here we modify the construction, in such a way that the resulting fractal can have **any** given box (or self-similar) dimension $0 < d < d_s = d_b$.

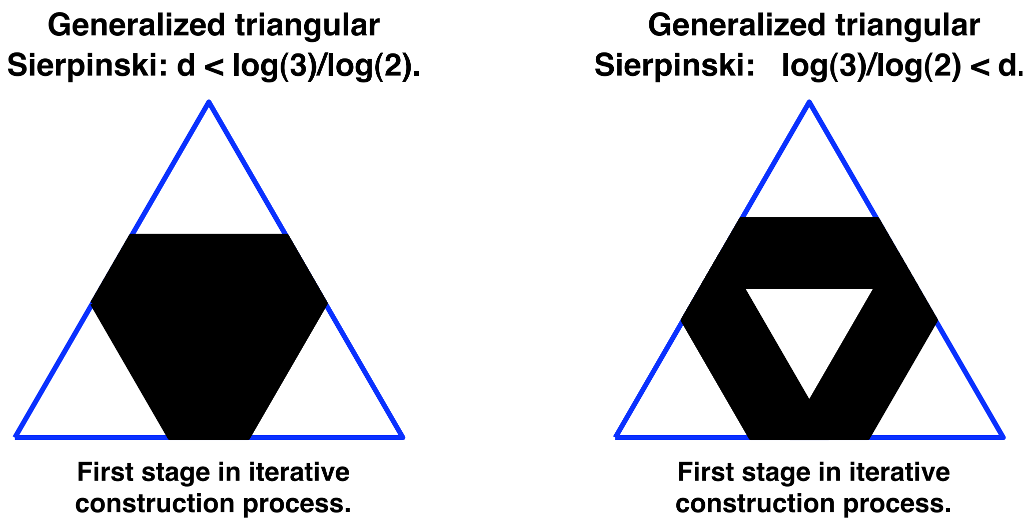

Figure 2.2: Generalizations of the Sierpinski gasket construction (details explained in the text). **Left:** First step in recursion defining a fractal with lower dimension than the Sierpinski gasket. Instead of removing just the center triangle, an extra "band" around it is also removed. The three remaining (equal and equilateral) triangles have linear dimensions reduced by some (fixed) factor $0 < s < 1/2$, relative to the starting triangle. **Right:** First step in recursion defining a fractal with higher dimension than the Sierpinski gasket. Instead of removing a whole chunk at the center of the starting triangle, only three bands (parallel to the sides) are removed. The four remaining (equal and equilateral) triangles have linear dimensions reduced by some (fixed) factor $0 < s < 1/2$, relative to the starting triangle.

The modified construction is illustrated on the left picture in figure 2.2. The change relative to the construction used in figure 2.1 is that, *in addition to removing the center triangle, an extra "band" around it is also removed.* This **band has width:**

$$\text{Width} = w\,h, \quad \text{where } 0 < w < \tfrac{1}{2} \tag{2.4}$$

is an arbitrary parameter (fixed throughout the construction), and h is the height of the starting triangle. What remains is still a set of three equal equilateral triangles, but their linear dimensions are reduced by a factor

$$0 < s = \frac{1}{2} - w < 0, \tag{2.5}$$

instead of $\frac{1}{2}$ (as before). It follows that the box dimension of the resulting fractal is

$$d = \lim_{n\to\infty} \frac{\log(3^n)}{-log(s^n L)} = -\frac{\log(3)}{log(s)} < d_b = d_s. \tag{2.6}$$

Notice also that (for this fractal) the self-similar dimension is equal to the box-dimension.

Part F: ***Generalizations with a higher dimension***

In this part we will modify the construction of the fractal, in such a way that the resulting fractal can be selected to have **any** given box (or self-similar) dimension $0 < d < 2$.

The modified construction is illustrated on the right picture in figure 2.2. The change relative to the construction used in figure 2.1 is that, *instead of removing a whole triangle at the center, only three "bands" (parallel to the sides) are removed.* Each **band has a width:**

$$\text{Width} = w\,h, \quad \text{where } 0 < w < \frac{2}{3} \tag{2.7}$$

is an arbitrary parameter (fixed throughout the construction), and h is the height of the starting triangle. One side of each band is a distance

$$\frac{1}{2}\left(1-\frac{1}{2}w\right)h \tag{2.8}$$

away from the corresponding opposite side, and the other side is a distance $s\,h$ away from the corresponding triangle vertex, where

$$0 < s = \frac{1}{4}(2-3w) < 1/2. \tag{2.9}$$

What remains is now a set of **four** equal equilateral triangles, with their linear dimensions reduced by a factor s. It follows that the box dimension for the resulting fractal is given by:

$$0 < d = \lim_{n\to\infty}\frac{\log(4^n)}{-log(s^n L)} = -\frac{\log(4)}{log(s)} < 2. \tag{2.10}$$

Notice also that (for this fractal) the self-similar dimension is equal to the box-dimension.

Part D: *The fractal's cardinal is equal to that of a square*

Here we show that the Sierpinski gasket has as many points as the unit square $\boldsymbol{S_q = \{0 < x,\, y < 1\}}$. The proof is not, strictly speaking, mathematically rigorous — it has (roughly) the same level of rigor as the proof in the book that the Cantor set has as many points as the unit interval (see examples 11.2.2 and 11.2.3 in pp. 403–404). For more details, see remark **2.2**.

We begin by showing[3] that

the unit interval $\boldsymbol{I_u = \{0 < x < 1\}}$ has as many points as the unit square $\boldsymbol{S_q}$.

Proof. For any point $u \in I_u$, consider its decimal representation: $u = 0.u_1\, u_2\, u_3\, u_4\, u_5 \ldots$, and use it to construct a correspondence with S_q via:

$$u \mapsto (x,\, y) = (0.u_1\, u_3\, u_5 \ldots,\; 0.u_2\, u_4\, u_6 \ldots). \tag{2.11}$$

The inverse map is obvious: merge the decimal expansions for x and y into one.

Next we show that

the points in an equilateral triangle can be represented by 4-adic expansions.[4]

That is, we can write

$$P = 0.d_1\, d_2\, d_3\, d_4\, d_5\, d_6 \ldots \quad \text{where } d_n \text{ is one of the digits: } 0,\ 1,\ 2, \text{ or } 3, \tag{2.12}$$

and P is any point in the triangle. To do this consider figure 2.1. Then: if P is in the middle triangle $d_1 = 0$, if P is in the top triangle $d_1 = 1$, if P is in the right triangle $d_1 = 2$, and if P is in the left triangle $d_1 = 3$. Next, in whichever sub-triangle P is in, the same process is repeated to determine[5] d_2, and so on.

It should be clear that: **in terms of the 4-adic representation above, the Sierpinski gasket is made by those points whose expansion does not include any zeros, i.e.: only the digits 1, 2, and 3 are used.**

Of course, we can re-interpret the three digit expansion (as above) for a point in the fractal, as a ternary representation of a point in the unit interval. It follows that

the fractal has as many points as the unit interval $\boldsymbol{I_u}$.

Since earlier we had shown that I_u and S_q had the same number of points, we also conclude that:

the fractal has as many points as the unit square $\boldsymbol{S_q}$.

[3] We need this because what we will show later is that the fractal has as many points as I_u.
[4] 4-adic means four digits.
[5] When the triangle being analyzed points down, instead of up, assign the digit 1 to the bottom sub-triangle.

Remark 2.2 ***Subtle issues and mathematical rigor.***
Earlier on it was mentioned that the proofs here are not quite mathematically rigorous. The reason is that there is a certain amount of ambiguity in the binary/ternary/etc. representations of real numbers (basically: there are numbers for which more than one representation is possible), and this creates some difficulties. In particular: the arguments above, which show one-to-one correspondences between the binary/ternary/etc. representations (but **not** *between the numbers themselves) have to be modified to account for this non-uniqueness. Sometimes this is simple, and sometimes it is not — but it always complicates the presentation, to the extent that the main idea becomes hard to follow.*

Examples of some of the issues involved:

— *To show that the Cantor set has as many points as the unit interval, the ternary representation of the points in the unit interval (employing only the digits 0, 1, and 2) is used. Then the members of the Cantor set are characterized as those not using the digit 1. But, for example, consider* $x = 0.0222\ldots$*, which can also be written as* $x = 0.1000\ldots$ *Is* x *in the Cantor set,*[6] *or is it not? Not a hard problem to fix, though it takes some writing to do so.*

— *Consider the argument showing that the unit square* S_q *and the unit interval* I_u *have the same number of points. Suppose now that we try to avoid trouble by using representations that do not end in an infinite string of 9's. But then, what do we do with numbers whose decimal representation has alternating 9's? Note, for example, that* $u = 0.19293959892949\ldots$*, yields (when the mapping in (2.11) is applied to it)* $x = 0.1235824\ldots$ *and* $y = 0.9999999\ldots$*!*

— *Consider the 4-adic representation for points in a triangle introduced in (2.12). If the point* P *happens to lie precisely at the boundary of two sub-triangles, ambiguity arises. What does this lead to? Well, it should be clear that, once a point shows up at one of the dividing lines (at some point in the process), at all subsequent stages the point will be on the perimeter of the triangle being analyzed — in fact, always on the "same" side. Thus the following is typical for the two possible expansions that arise:*

$$P = 0.1\, d_1\, d_2\, d_3 \cdots = 0.0\, d_1\, d_2\, d_3 \ldots,$$

where the d_n*'s are all 2's or 3's. In other words: expansions that end with an infinite string involving only two of the digits 1, 2, or 3, give rise to ambiguity in the 4-adic representation. As in the case of other representations of this kind, the problem arises at the boundaries of the dividing regions used to produce the expansion. If no rule is given as to how to "pick sides", ambiguity arises.*

Remark 2.3 ***Fractal definition details.***
When we defined the fractal, we did not specify if the triangle being removed was supposed to be removed with, or without, its boundaries. None of the earlier arguments in this problem are affected by this, though note that (in fact) each choice gives rise to a **different** *fractal.*
An interesting point is that, if we choose to remove only open triangles, then showing that the fractal has as many points as the unit square is straightforward. Why? Because in this case the triangle's sides are never removed, hence the fractal has (at least) as many points as an interval. On the other hand, it clearly has less points than a square (since it is a subset of any square large enough to include the original triangle). But a square and an interval have the same number of points, so the fractal must also. Of course: all of this holds only if it turns out that the usual rules for bigger than, smaller than, and equality, apply to cardinals for infinite sets … which they (mostly) do, but I have not shown this (nor will I).

Remark 2.4 ***Convergence, completeness, etc.***
If you did not fall asleep when the 4-adic expansion for points in a triangle was introduced in (2.12), the following two questions might have occurred to you:

[6] This is related to the question of whether, when removing the middle third, the ends are removed or not.

$\mathbf{Q_1}$. If I take two different points P_1 and P_2, are their 4-adic expansions different? *The answer is:* **yes,** *and the proof is trivial: Because the points are different, they are a finite distance away. Then, because the sub-triangles the starting triangle is subdivided into keep on getting smaller, eventually the two points end up in different sub-triangles and their 4-adic expansions turn out not equal.*
$\mathbf{Q_2}$. If I just write an arbitrary 4-adic expansion, is there a point on the triangle that gives it? *Again, the answer is* **yes,** *the proof as follows: What the 4-adic expansion actually does is to describe a sequence of nested sub-triangles inside the starting triangle, whose length scale goes down by a factor of 2 at every stage*[7] *Thus: consider the* sequence made up by the centers *of these sub-triangles. It should be clear that this is a* Cauchy sequence, *and so it has a limit. This limit is the point in the starting triangle with the desired properties.*

The arguments above answer the question:
What is needed to generate a p-adic (p digits) representation of the points in a set?

1. *A rule for splitting the set (and each of the resulting sub-sets) into p non-empty parts.*
2. *A rule for naming the p parts of each split.*
3. *The size of the parts must go to zero as the number of splits goes to infinity.*
4. *The starting set must be* **complete.** *That is, every Cauchy sequence in it must have a limit.*

By the way: Notice that this (for example) gives a scheme for generating, and keeping track of the elements in general, non-cartesian, numerical grids, *so this is not completely idle speculation.*

Remark 2.5 ***Extent of non-uniqueness.***
Earlier on we mentioned that the 4-adic expansion for points in a triangle introduced in (2.12) is not unique, in the sense that a point may have more than one such expansion. **How bad is this problem? The answer is: not that bad,** *as we show next. The only points with problems are the ones that land (at same stage) on the boundary between two sub-triangles. At that stage there is a choice of which sub-triangle to use, but afterwards the point ends up on the outside edge of all subsequent sub-divisions, so no more non-uniqueness arises — with one exception: points that land at a corner where three sub-triangles meet. In this case three choices are possible, but afterwards there is no more nonuniqueness, for in subsequent sub-divisions the point will be at an outside vertex of the triangle being divided. Thus, it is clear that the answer to the question of non-uniqueness is as follows:*

Some points have two possible 4-adic expansions. *These are of the following three types:*

1. *Up to some place both expansions are the same, and:*
From then on one of them has the form $\{0\, d_1\, d_2\, d_3\, d_4 \dots\}$ *and the other* $\{1\, d_1\, d_2\, d_3\, d_4 \dots\}$, *where the* d_n *are all either 2 or 3. Note that the difference between the two expansion is just one digit!*
2. *Up to some place both expansions are the same, and:*
From then on one of them has the form $\{0\, d_1\, d_2\, d_3\, d_4 \dots\}$ *and the other* $\{2\, p_1\, p_2\, p_3\, p_4 \dots\}$, *where the digits* d_n *are all either 1 or 2, and* $p_n = 1$ *if* $d_n = 2$, *or* $p_n = 3$ *if* $d_n = 1$.
3. *Up to some place both expansions are the same, and:*
From then on one of them has the form $\{0\, d_1\, d_2\, d_3\, d_4 \dots\}$ *and the other* $\{3\, p_1\, p_2\, p_3\, p_4 \dots\}$, *where the digits* d_n *are all either 1 or 3, and* $p_n = 1$ *if* $d_n = 3$, *or* $p_n = 2$ *if* $d_n = 1$.

Some points have three possible 4-adic expansions. *These are of the following three types:*

1. *Up to some place both expansions are the same. From then on they continue with one of the three following choices:* $\{0\,2\,2\,2\,2 \dots\}$, $\{1\,2\,2\,2\,2 \dots\}$, *or* $\{2\,1\,1\,1\,1 \dots\}$.
2. *Up to some place both expansions are the same. From then on they continue with one of the three following choices:* $\{0\,1\,1\,1\,1 \dots\}$, $\{2\,3\,3\,3\,3 \dots\}$, *or* $\{3\,2\,2\,2\,2 \dots\}$.

[7] That is, in the sequence, every sub-triangle is half the size, and inside, the preceding one.

3. *Up to some place both expansions are the same. From then on they continue with one of the three following choices:* $\{0\,3\,3\,3\,3\ldots\}$, $\{1\,3\,3\,3\,3\ldots\}$, *or* $\{3\,1\,1\,1\,1\ldots\}$.

The ***basic reason none of this non-uniqueness makes a dent on the cardinality arguments,*** *is that it arises only with a finite factor. Worse case scenario: imagine that* every *point has (say) three possible representations. Then the set of all possible p-adic expansions would be three times as large as the starting set. However, 3 times infinity (or 2 times infinity, or 10 times infinity) gives you back the same infinity! — Examples: (a) The intervals* $\{0 < x < 1\}$ *and* $\{0 < x < 3\}$ *have the same number of points. (b) The set of natural numbers and the set of integers have the same number of points. For that matter: the square of an infinity (or its cube, or ..) is again the same infinity — Examples: (a) The unit square has as many points as the unit interval. (b) The rational numbers have as many points as the natural numbers. The fact is:* The arithmetic of infinity is very strange!

A final point: To be honest, I do not think that any of this stuff about dimensions and cardinality is particularly important. But it is an interesting game.

3 Problem 08.06.03 - Strogatz (Irrational flow yields dense orbits)

3.1 Statement for problem 08.06.03

Consider the flow in the torus given by

$$\frac{d\theta_1}{dt} = \omega_1 \quad \text{and} \quad \frac{d\theta_2}{dt} = \omega_2, \tag{3.1}$$

where ω_1 and ω_2 are non-zero constants, θ_1 and θ_2 are angles (thus we identify any two values that differ by a multiple of 2π) and $\mu = \omega_1/\omega_2$ is *irrational*. Show that each trajectory is **dense**; i.e., given any point p on the torus, any initial condition q, and any $\epsilon > 0$, there is some $t < \infty$ such that the trajectory starting at q passes within a distance ϵ of p.

Hint #1. The lemma below provides a key step in the argument. **Prove it.**

Lemma: *Let* $\mathcal{Z}$ *be the set of all the integers and let* μ *be an irrational number.*
Then $2\,\pi\,\mu\,\mathcal{Z}\ (\mathrm{mod}\,2\,\pi)$ *is dense in* $[0,\,2\,\pi)$.

Note: Two numbers are equal mod $2\,\pi$ if their difference is an integer multiple of $2\,\pi$ — i.e.: think of the numbers as angles in radians. In particular, think of $2\,\pi\,\mu\,\mathcal{Z}\ (\mathrm{mod}\,2\,\pi)$ as the set of all the numbers $0 \leq x < 2\,\pi$ which are equal mod $2\,\pi$ to a number of the form $2\,\pi\,\mu\,n$, with $n \in \mathcal{Z}$.

Hint #2. Show that the numbers in $2\,\pi\,\mu\,\mathcal{Z}\ (\mathrm{mod}\,2\,\pi)$ *are all different. Since they are all in* $0 \leq x < 2\,\pi$*, what happens with the distances between them? Exploit this and the two equalities:* $x_p - x_q = x_{p-q}$ *and* $x_{p+jk} = x_p + j\,x_k$*, which (of course) also apply* mod $2\,\pi$.

Hint #3. Identify the Torus with the rectangle $\mathcal{R} = [0,\,2\,\pi) \times [0,\,2\,\pi)$*, and (for any orbit) consider the set produced by its intersections with the bottom side of* $\mathcal{R}$ *— that is* $\theta_2 = 0\ (\mathrm{mod}\,2\,\pi)$.

3.2 Answer for problem 08.06.03

To show that **the trajectories given by flow in (3.1) are dense,** we begin by proving the lemma.

Proof: Let $\{x_n\}$ be the set of numbers in $[0,\, 2\,\pi)$ defined by $x_n = 2\,\pi\,n\,\mu\ (\mathrm{mod}\ 2\,\pi)$, where $n \in \mathcal{Z}$. **These numbers are all distinct:** $x_n = x_m$ for $n \neq m$ implies $2\,\pi\,(n-m)\,\mu = 0\ (\mathrm{mod}\ 2\,\pi)$ — i.e.: $2\,\pi\,(n-m)\,\mu = 2\,\pi\,j$ for some $j \in \mathcal{Z}$, which cannot be since μ is irrational.

For any natural number N, consider the set $\{x_1,\, x_2,\, \dots\, x_N\}$ in $[0,\, 2\,\pi)$. At least two of these numbers must be within a distance $2\,\pi/N$ of each other: there exist $1 \leq p < q \leq N$ such that $0 < |x_p - x_q| \leq 2\,\pi/N$. But $x_{j(q-p)} = j\,(x_q - x_p)\ (\mathrm{mod}\ 2\,\pi)$ for any integer j. Thus the numbers $x_{j(q-p)}$ are separated by a distance less than $2\,\pi/N$. Hence, within a distance $2\,\pi/N$ of any point in $[0,\, 2\,\pi)$ there is a point in the set $\{x_n\}$. Since N is arbitrary, it follows that $\{x_n\}$ is dense in $[0,\, 2\,\pi)$. ♣

Now back to the original problem. It is clear that the orbits for (3.1) have the form:

$$\theta_1 = \mu\,\tau + \phi\ (\mathrm{mod}\ 2\,\pi) \quad \text{and} \quad \theta_2 = \tau\ (\mathrm{mod}\ 2\,\pi), \tag{3.2}$$

where $-\infty < \tau < \infty$ and $0 \leq \phi < 2\pi$ is a constant — where we identify the Torus with the rectangle $\mathcal{R} = [0,\, 2\,\pi) \times [0,\, 2\,\pi)$. Pick now an arbitrary orbit, as given by (3.2), and consider its intersections with the bottom side of the rectangle $\mathcal{R}$. That is, with the set: $\{0 \leq \theta_1 < 2\,\pi,\, \theta_2 = 0\}$. This means that $\tau = 2\,\pi\,n$, $n \in \mathcal{Z}$, in (3.2). The set of θ_1's that follow is $2\,\pi\,\mu\,\mathcal{Z} + \phi\ (\mathrm{mod}2\pi)$, which (from the lemma) is dense in $[0,\, 2\,\pi)$.

Think now of the geometry involved (see figures 8.6.4, page 275, and figure 8.6.6, page 276, in the book): in $\mathcal{R}$, the orbit is the set of lines with slopes $1/\mu$ going through one of the points in the set $\{\theta_2 = 0,\ \ \theta_1 \in 2\,\pi\,\mu\,\mathcal{Z} + \phi\ (\mathrm{mod}2\pi)\}$. The distance between any two of these lines is no greater than the distance between the corresponding points in $2\,\pi\,\mu\,\mathcal{Z} + \phi\ (\mathrm{mod}2\pi)$. It should then be quite obvious that the orbit is dense.

4 Problem 08.06.05 - Strogatz (Plotting Lissajous figures)

4.1 Statement for problem 08.06.05

Using a computer, plot the curve whose parametric equations are $x(t) = \sin(t)$ and $y(t) = \sin(\omega\,t)$, for the following rational and irrational values of the parameter ω.

$$\left.\begin{array}{lll} \textbf{(a)}\ \ \omega = 3 & \textbf{(c)}\ \ \omega = \dfrac{2}{3} & \textbf{(c)}\ \ \omega = \dfrac{5}{3} \\ \textbf{(b)}\ \ \omega = \sqrt{2} & \textbf{(d)}\ \ \omega = \pi & \textbf{(f)}\ \ \omega = \dfrac{1}{2}\left(1+\sqrt{5}\right) \end{array}\right\} \tag{4.1}$$

The resulting curves are called *Lissajous* figures. They can be displayed on an oscilloscope by using two ac signals of different frequencies as inputs.

Remark 4.6 *Obviously, in a numerical calculation ω cannot be "irrational". However, an irrational ω will be approximated by a rational number which is the quotient of two very large integers. The result will be a curve with a very, very, complicated and a very, very long period.*

4.2 Answer for Problem 8.6.5

Use the MatLab script Lissajous(omega, N) to calculate and plot the Lissajous figures for arbitrary values of ω. In particular, **for ω irrational,** take N large to see the **space filling property of the curve.**

5 Problem 09.02.02 - Strogatz. Ellipsoidal trapping region for the Lorenz attractor

5.1 Statement for problem 09.02.02

Consider the Lorenz system of equations:

$$\frac{dx}{dt} = \sigma\,(y - x), \quad \frac{dy}{dt} = r\,x - y - x\,z, \qquad \text{and} \qquad \frac{dz}{dt} = x\,y - b\,z, \tag{5.1}$$

where σ, r and b are **positive** constants. Let Ω be the ellipsoidal region given by the points satisfying the equation

$$E = r\,x^2 + \sigma\,y^2 + \sigma\,(z - 2\,r)^2 \leq C, \tag{5.2}$$

where $C > 0$ is a constant. **Show that,** for C large enough, all trajectories of the Lorenz equations eventually enter Ω and stay there forever. *Hint: compute* $\frac{\boldsymbol{dE}}{\boldsymbol{dt}}$.

For a much stiffer challenge: try to obtain the smallest possible value of C with this property.

5.2 Answer for problem 09.02.02

For E as in (5.2), we have: $$\frac{dE}{dt} = -2\,\sigma\,(\Phi - b\,r^2), \quad \text{where} \quad \Phi = r\,x^2 + y^2 + b\,(z - r)^2.$$

It follows that, if we take C large enough so that the ellipsoid $E = C$ includes the ellipsoid $\Phi = b\,r^2$ inside it, then $\dot{E} \leq -\delta < 0$ (for some $\delta > 0$) for $E \geq C$, which proves the desired result.

$$\textbf{Notation: } \boldsymbol{\mathcal{E}_C} \text{ is the ellipsoid } E = C, \text{ and } \boldsymbol{\Gamma} \text{ is the ellipsoid } \boldsymbol{\Phi = b\,r^2}. \tag{5.3}$$

To find the smallest possible value that $\boldsymbol{C}$ can take, we note that the ellipsoids $\mathcal{E}_C$ are nested. Let $\boldsymbol{C_m}$ *be the maximum value that* $\boldsymbol{E}$ *takes over the ellipsoid* $\boldsymbol{\Gamma}$. Then $\boldsymbol{\Gamma}$ *is inside*[8] $\boldsymbol{\mathcal{E}_m = \mathcal{E}_{C_m}}$ (touching $\mathcal{E}_m$ at the points where $E = C_m$ over Γ) and $E \leq C_m$ over Γ. Then $C > C_m$ yields $\mathcal{E}_C$ which properly includes Γ, so that $\frac{dE}{dt} < 0$ over $\mathcal{E}_C$.

$$\text{Thus } \boldsymbol{C_m} \textbf{ is the value we are looking for.} \tag{5.4}$$

Strictly speaking there is no smallest value C such that $\frac{dE}{dt} < 0$ over $\mathcal{E}_C$, but C_m is the infimum of all such values.

To compute $\boldsymbol{C_m}$ we proceed as follows: Note that the value of z for Γ is constrained to the range $0 \leq z \leq 2\,r$. Thus we can parameterize Γ as follows:

$$x = \frac{1}{\sqrt{r}}\,\rho\,\cos(\theta), \quad y = \rho\,\sin(\theta), \quad \text{and} \quad z = 2\,r - t,$$

where $\quad 0 \leq \theta < 2\,\pi, \quad 0 \leq t \leq 2\,r, \quad$ and $\quad \rho = \sqrt{b\,r^2 - b\,(z - r)^2} = \sqrt{2\,b\,r\,t - b\,t^2}$.

Then

$$\boldsymbol{E = \rho^2\,\cos^2(\theta) + \sigma\,\rho^2\,\sin^2(\theta) + \sigma\,t^2}. \tag{5.5}$$

To find the maximum of this expression over the given parameter range, we consider two cases:

- **Case $\boldsymbol{\sigma \geq 1}$**. Then a maximum requires $\theta = \pi/2,\ 3\pi/2$, where $\boldsymbol{E = \sigma\,(\rho^2 + t^2) = \sigma((1 - b)\,t^2 + 2\,b\,r\,t)}$.

  Hence: (1) if $\boldsymbol{b \leq 2}$, $\boldsymbol{C_m = 4\,\sigma\,r^2}$, and (2) if $\boldsymbol{b \geq 2}$, $\boldsymbol{C_m = \dfrac{\sigma\,b^2\,r^2}{b - 1}}$.

- **Case $\boldsymbol{\sigma \leq 1}$**. Then a maximum requires $\theta = 0,\ \pi$, where $\boldsymbol{E = \rho^2 + \sigma\,t^2 = (\sigma - b)\,t^2 + 2\,b\,r\,t}$.

  Hence: (1) if $\boldsymbol{b \leq 2\,\sigma}$, $\boldsymbol{C_m = 4\,\sigma\,r^2}$, and (2) if $\boldsymbol{b \geq 2\,\sigma}$, $\boldsymbol{C_m = \dfrac{b^2\,r^2}{b - 1}}$.

[8] This follows because of the nesting and the fact that $E \leq C_m$ over Γ.

Putting it all together, the answer is:

- $\sigma \geq 1$ and $b \leq 2$ .... $C_m = 4\,\sigma\,r^2$.
- $\sigma \geq 1$ and $b \geq 2$ .... $C_m = \dfrac{\sigma\,b^2\,r^2}{b-1}$.
- $\sigma \leq 1$ and $b \leq 2\,\sigma$ .... $C_m = 4\,\sigma\,r^2$.
- $\sigma \leq 1$ and $b \geq 2\,\sigma$ .... $C_m = \dfrac{b^2\,r^2}{b-\sigma}$.

6 Problem 09.06.02 - Strogatz (Pecora and Carroll's approach)

6.1 Statement for problem 09.06.02

Pecora and Carroll's approach for signal transmission/reception using the Lorenz system. In the pioneering work of Pecora and Carroll[9] one of the receiver variables is simply set *equal* to the corresponding transmitter variable. For instance, if $x(t)$ is used as the transmitter drive signal, then the receiver equations are

$$\left.\begin{array}{rcl} x_r(t) & \equiv & x(t), \\ \dfrac{dy_r}{dt} & = & r\,x(t) - y_r - x(t)\,z_r, \\ \dfrac{dz_r}{dt} & = & x(t)\,y_r - b\,z_r, \end{array}\right\} \tag{6.1}$$

where the first equation is **not** a differential equation.[10] Their numerical simulations, and a heuristic argument, suggested that $y_r(t) \to y(t)$ and $z_r(t) \to z(t)$ as $t \to \infty$, even if there were differences in the initial conditions.

Here are the steps for simple proof of the result stated above, due to He and Vaidya.[11]

A. Show that the error dynamics are governed by:

$$\left.\begin{array}{rcl} e_x(t) & \equiv & 0, \\ \dfrac{de_y}{dt} & = & -e_y - x(t)\,e_z, \\ \dfrac{de_z}{dt} & = & x(t)\,e_y - b\,e_z, \end{array}\right\} \tag{6.2}$$

where $\boldsymbol{e_x = x - x_r}$, $\boldsymbol{e_y = y - y_r}$, and $\boldsymbol{e_z = z - z_r}$.

B. Show that $\boldsymbol{V = (e_y)^2 + (e_z)^2}$ is a Liapunov function.

C. What do you conclude?

[9] Pecora, L. M., and Carroll, T. L., *Synchronization in chaotic systems.* Phys. Rev. Lett. **64:**821, (1990).

[10] This equation replaces the first equation $\dot{x_r} = \sigma\,(y_r - x_r)$ in a Lorenz system for $(x_r,\,y_r,\,z_r)$. Then x is used to replace x_r in the other two equations. The Lorenz system constants are σ, r, b.

[11] He, R., and Vaidya, P. G., *Analysis and synthesis of synchronous periodic and chaotic systems.* Phys. Rev. A, **46:**7387 (1992).

6.2 Answer for problem 09.06.02

A. The Lorenz system is given by

$$\left.\begin{aligned}\frac{dx}{dt} &= \sigma\,(y-x),\\ \frac{dy}{dt} &= r\,x - y - x\,z,\\ \frac{dz}{dt} &= x\,y - b\,z.\end{aligned}\right\} \tag{6.3}$$

where σ, r and b are **positive** constants. Thus, subtracting the last two equations in (6.1), from the last two here, the system in (6.2) follows — the first equation in (6.2) is obvious, since x is the same in both receiver and transmitter.

B. Next we note that

$$\frac{dV}{dt} = -2\,(e_y)^2 - 2\,b\,(e_z)^2, \tag{6.4}$$

from which it follows that V **is a Liapunov function**.

C. It follows that, **for any initial conditions,** $\boldsymbol{V \to 0}$ **as** $\boldsymbol{t \to 0}$**.** That is: $\boldsymbol{e_y \to 0}$ and $\boldsymbol{e_z \to 0}$ as $\boldsymbol{t \to 0}$, which is precisely what we wanted to show.

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7 Problem 14.10.03 - Newton's method in the complex plane

7.1 Statement for problem 14.10.03

Suppose that you want to solve an equation, $g(x) = 0$. Then you can use *Newton's method*, which is as follows: Assume that you have a "reasonable" guess, x_0, for the value of a root. Then the sequence $\boldsymbol{x_{n+1} = f(x_n)}$, $\boldsymbol{n \geq 0}$, where

$$\boldsymbol{f(x) = x - \frac{g(x)}{g'(x)}}, \tag{7.1}$$

converges (very fast) to the root.

Remark 7.7 (The idea). *Assume an approximate solution $g(x_a) \approx 0$. Write $x_b = x_a + \delta x$ to improve it, where δx is small. Then $0 = g(x_a + \delta x) \approx g(x_a) + g'(x_a)\,\delta x \;\Rightarrow\; \delta\,x \approx -\frac{g(x_a)}{g'(x_a)}$, and (7.1) follows.*
Of course, if x_0 is not close to a root, the method may not converge. Even if it converges, it may converge to a root that is far away from x_0, not necessarily the closest root. In this problem we investigate the behavior of Newton's method in the complex plane, *for arbitrary starting points.*

Consider iterations of the map in the complex plane generated by Newton's method for the roots of $z^3 - 1 = 0$. That is

$$z_{n+1} = f(z_n) = \left(\frac{2}{3} + \frac{1}{3\,z_n^3}\right) z_n, \quad n \geq 0, \tag{7.2}$$

where $0 < |z_0| < \infty$ is arbitrary. Note that

$$\boldsymbol{\zeta_1 = 1, \quad \zeta_2 = e^{i\,2\,\pi/3} = \frac{1}{2}\,(-1 + i\,\sqrt{3}),} \quad \text{and} \quad \boldsymbol{\zeta_3 = e^{i\,4\,\pi/3} = \frac{1}{2}\,(-1 - i\,\sqrt{3}),} \tag{7.3}$$

are the roots of $z^3 = 1$.

Your tasks: Write a computer program to calculate the orbits $\{z_n\}_{n=0}^{\infty}$. Then, for every [12] initial point z_0, draw a colored dot at the position of z_0, where the colors are picked as follows:

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[12] Numerically this means: choose a sufficiently fine grid in a rectangle, and pick every point in the grid. For example, select the square $-2 < x < 2$ and $-2 < y < 2$, where $z_0 = x + i\,y$.

$z_n \to \zeta_1$, cyan. $\quad z_n \to \zeta_2$, magenta. $\quad z_n \to \zeta_3$, yellow. $\quad$ No convergence, black. (7.4)

What do you see? Do blow ups of the limit regions between zones.

Hint. Deciding that the sequence converges is easy: once z_n gets "close enough" to one of the roots, then the very design of Newton's method guarantees convergence. Thus, given a z_0, compute z_N for some large N, and check if $|z_N - \zeta_j| < \delta$ for one of the roots and some "small" tolerance δ — which does not have to be very small, in fact $\delta = 0.25$ is good enough. You can get pretty good pictures with $N = 50$ iterations on a 150×150 grid. A larger N is needed when refining near the boundary between zones.

Hint. If you use MatLab, do not plot "points". Instead, plot "regions", where the color of each pixel is decided by z_0 — use the command image(x, y, C) *to plot. Why? Because using points leaves a lot of unpainted space in the figure, and gives much larger file sizes.*

7.2 Answer for problem 14.10.03

Figures 7.1 and 7.2 show the results of our calculations. Note the *fractal nature of the boundary between the basins*

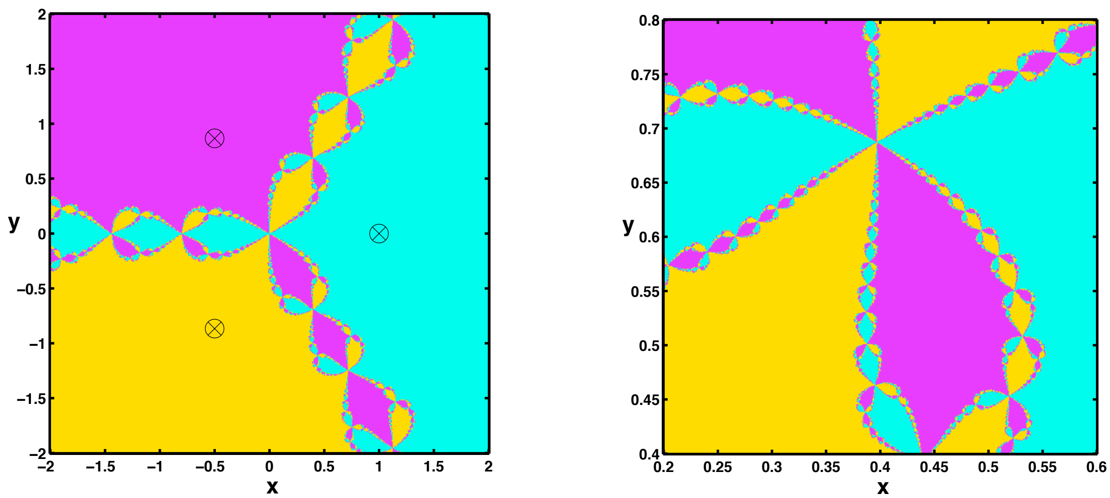

Figure 7.1: (Problem 14.10.03). Convergence zones for the $z^3 = 1$ Newton's map iterates. Color scheme in (7.4), with a 500×500 pixel grid. Left: $N = 100$ iterations, for $-2 < x, y < 2$. The crosses are the roots ζ_j. Right: $N = 200$ iterations, for $0.2 < x < 0.6$ and $0.4 < y < 0.8$.

of attraction for each root: as we zoom in, the object appears as a smaller (but distorted) copy of itself. Non-trivial self-similarity [13] is the hallmark of a fractal. Sets like this (boundaries between convergence regions of complex analytic iterations) are called **Julia sets**.

The attracting basins are **Fatou sets**. The sets are named after Gaston Julia and Pierre Fatou, two mathematicians that pioneered the study of complex dynamics — e.g., see: G. Julia, *Mémoire sur l'iteration des fonctions rationnelles,* Journal de Mathématiques Pures et Appliquées, **8:** 47–245, 1918, and P. Fatou, *Sur les substitutions rationnelles*, Comptes Rendus de l'Académie des Sciences de Paris, **164:** 806–808 and vol. 165, pp. 992–995, (1917).

The orbits within the Julia set are chaotic. These orbits are, generally, not periodic (but recurrent), and small differences in z_n grow exponentially with n (sensitive dependence on initial conditions). However, computing these orbits is extremely hard, as perturbations out of the Julia set make the resulting orbit convergent.

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[13] A line in the plane is also self-similar, but it has trivial structure.

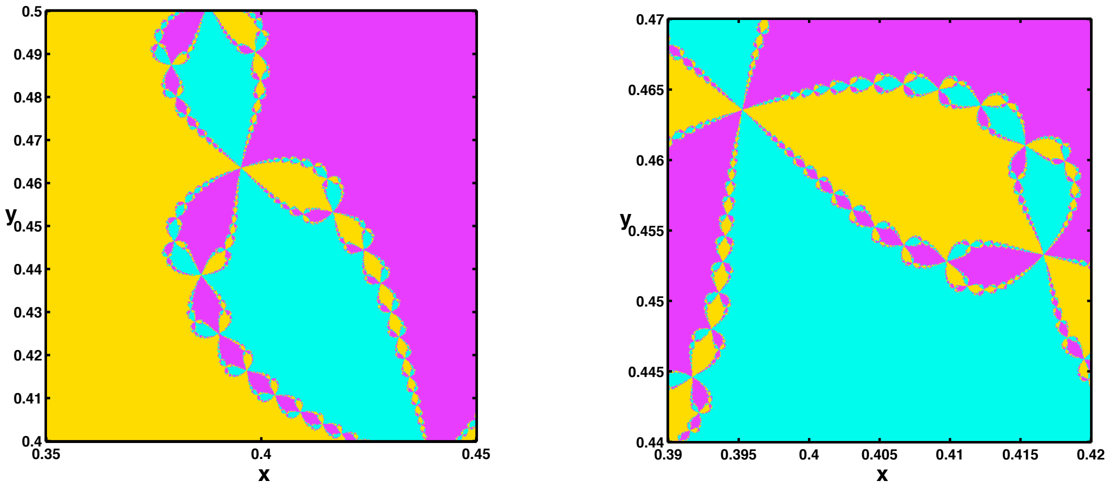

Figure 7.2: (Problem 14.10.03). See figure 7.1. Further blow ups with $N = 300$ iterations. Left: region $0.35 < x < 0.45$ and $0.4 < y < 0.5$. Right: region $0.39 < x < 0.42$ and $0.44 < y < 0.47$.

8 Problem 11.03.08 - Strogatz. Sierpinski's carpet

8.1 Statement for problem 11.03.08

Consider the process shown in figure 8.1. The closed unit box is divided into nine equal boxes, and the open central box is deleted. Then this process is repeated for each of the eight remaining sub-boxes, and so on. Figure 8.1 shows the first two stages.

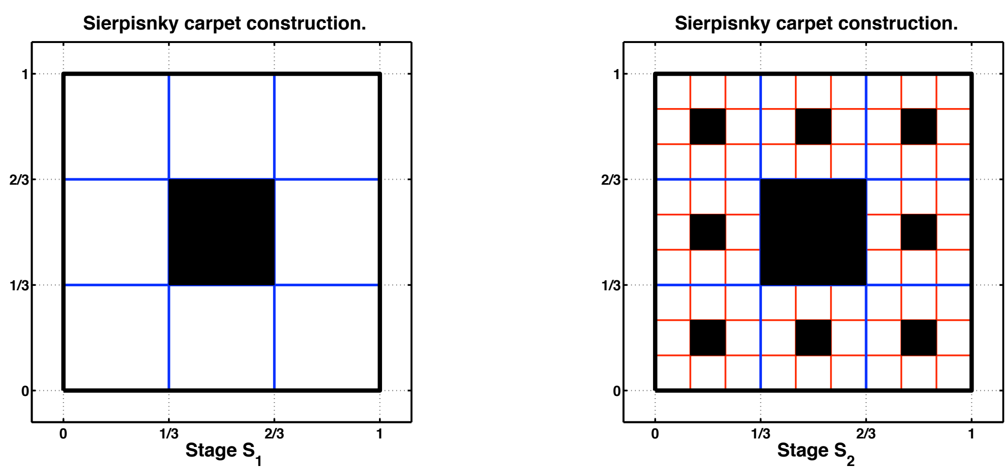

Figure 8.1: Problem 11.3.8. **Sierpinski carpet construction.** The areas shaded in black are the parts of the original square deleted at each stage of the fractal's construction.

A. Sketch the next stage, S_3.

B. Find the similarity dimension of the limiting fractal, known as the **Sierpinski carpet.**

C. Show that the Sierpinski carpet has zero area.

8.2 Answer for problem 11.03.08

A. Figure 8.2 shows stages S_3 and S_4 in the fractal construction.

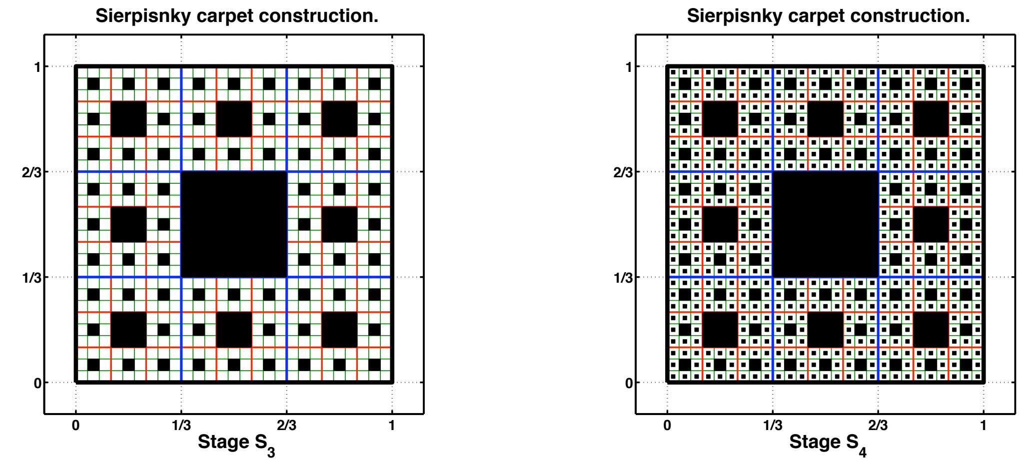

Figure 8.2: Problem 11.3.8. Sierpinski carpet construction. Left: Stage S_3. Right: stage S_4. For this last stage we have not drawn the dividing lines at the last level, to avoid cluttering the picture.

B. The carpet is made up by 8 identical copies of itself, each one reduced in size by a factor 3. In fact, for any $n = 1, 2, 3, 4, \ldots$, *the fractal is made up by* $N = 8^n$ *identical copies of itself, each one reduced in size by a factor* $r = 3^n$. Thus, the **self-similarity dimension** d **of the fractal** is given by:

$$d = \frac{\log(N)}{\log(r)} = \frac{\log(8^n)}{\log(3^n)} = \frac{\log(8)}{\log(3)} \approx 1.8928.$$

C. Let A_0 be the area of the box from which the fractal construction is started, and A_n the area of the set S_n. Clearly: $A_{n+1} = \frac{8}{9} A_n$, so that $A_n = \left(\frac{8}{9}\right)^n A_0$. Since the fractal is included in **all** the S_n, and $A_n \to 0$ as $n \to 0$, it follows that **the fractal has zero area.**

THE END.