Answers to P-Set # 05, 18.385j/2.036j
MIT (Fall 2020)

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1.1 Statement: Large μ limit for Liénard system #03

A Liénard equation has the form
$$\ddot{x} + \mu\, f'(x)\, \dot{x} + g(x) = 0, \tag{1.1}$$
for some functions f and g. Here $\mu > 0$ is a parameter.

This can be re-written in the form
$$\frac{d}{dt}\,(\dot{x} + \mu\, f(x)) + g(x) = 0. \tag{1.2}$$
Introduce $y = \frac{1}{\mu}\,\dot{x} + f(x)$, to get the system
$$\dot{x} = \mu\,(y - f(x)) \quad \text{and} \quad \dot{y} = -\frac{1}{\mu}\, g(x). \tag{1.3}$$

In this problem we will consider the case
$$f(x) = -x + \frac{1}{3}\,x^3 - \frac{1}{60}\,x^5 \quad \text{and} \quad g(x) = x, \tag{1.4}$$
with $\boldsymbol{\mu \gg 1}$.

Analyze the large μ limit for this system. In particular:

1. Are there any limit cycles? Are they stable, unstable, semi-stable?
2. Are there any critical points? Are they attractors, repellers?
3. Does the system have any global attractor?
4. Sketch the phase plane portrait.

1.2 Answer: Large μ limit for Liénard system #03

The behavior in the $\mu \gg 1$ limit is controlled by the following facts:

f01. $g(x) > 0$ for $x > \boldsymbol{x_g = 0}$, and $g(x) < 0$ for $x < x_g$.

Thus $\boldsymbol{\dot{y} < 0}$ **for** $\boldsymbol{x > x_g}$ **and** $\boldsymbol{\dot{y} > 0}$ **for** $\boldsymbol{x < x_g}$.

Let $\boldsymbol{y_g = f(x_g) = 0}$.

f02. The curve $y = f(x)$ (see the dashed black line in figure 1.1) has four special points $\boldsymbol{\{x_n\}_{n=1 \text{ to } 4}}$, with corresponding values $\boldsymbol{y_n}$, given by its local maximums/minimums. We have

1) $x_1 < x_2 < x_g < x_3 < x_4$.
2) x_1 and x_3 are local minimums, with $y_1 < y_3$.
3) x_2 and x_4 are local maximums, with $y_2 < y_4$.
4) The curve is (strictly) monotonic in each of the intervals open intervals $I_n = (x_n,\, x_{n+1})$, n=0:4, where $x_0 = -\infty$ and $x_5 = \infty$ — decreasing, increasing, decreasing, ...
5) $y \to \pm\infty$ as $x \to \mp\infty$.

Notation: The **branch I_n** means the portion of the curve given by $y = f(x)$, $x \in I_n$.

Notation, for points: $\boldsymbol{\mathcal{P}_n = (x_n,\, y_n)}$ and $\boldsymbol{\mathcal{C} = (x_g,\, y_g)}$ **= critical point.**

$\boldsymbol{\mathcal{C}}$ **is the only critical point.**

f03. Since $\mu \gg 1$, y changes very slowly, while x changes very fast (except when $y - f(x)$ is small).

It follows that:

f04. For $y - f(x) > 0$ (but not too small), the phase plane trajectories are nearly horizontal lines, with the flow left to right.

f05. For $y - f(x) < 0$ (but not too small), the phase plane trajectories are nearly horizontal lines, with the flow right to left.

f06. In the **t growing direction** the trajectories get trapped in a narrow band near the branches I_1 and I_3 — which they can exit only near the points P_2 and P_3, see item **f08**.
These are the **stable/attracting** branches.

f07. In the **t decreasing direction** the trajectories get trapped in a narrow band near the branches I_0, I_2, and I_4 — which they can "exit" only near the points P_1 for I_0, P_2 for the left half of I_2, P_3 for the right half of I_2, and P_4 for I_4, see item **f08**.
These are the **unstable/repelling** branches.
Hint: the easy way to figure out the phase portrait near an unstable branch, is to draw it for the reversed flow (t decreasing), and then reverse the arrows.

f08. In the **trapping narrow band near each branch**, the solutions slowly move up or down, depending on the sign of g. That is:

Up for I_0, I_1, and the left side ($x < x_g$) of I_2.
Down for I_4, I_3, and the right side ($x > x_g$) of I_2.

Eventually (forward/backwards in time for the stable/unstable branches) the solutions get pushed past one of the points P_n. At this point the trajectories "exit" the narrow trapping band, and the trajectories switch to the behavior in either item **f04** or item **f05**.

Task #1 left to the reader. *How do the solutions behave in the trapping bands near the branches? In particular: Do the solutions cross the branches? If so, which solutions, and how is the crossing? Is the narrow trapping band on one side of the branch (which side?), or does it enclose the branch?*

Task #2 left to the reader. *How do the solutions behave near the points P_n, n=1:4? Sketch a detail of the phase portrait near these points.*

f09. From the facts above, it follows that all the orbits leave the critical point $\mathcal{C}$ along the narrow band near I_2, up to the left, or down to the right. This with two exceptions: the nearly horizontal orbits that leave $\mathcal{C}$ to the left, or to the right (these two orbits are the transition from the orbits leaving up to the left, and the ones leaving down to the right). It follows that **$\mathcal{C}$ is an unstable node.**

Task #3 left to the reader. *How do the solutions behave near the critical point $\mathcal{C}$? Sketch a detail of the phase portrait near this point, including the branch I_2.*

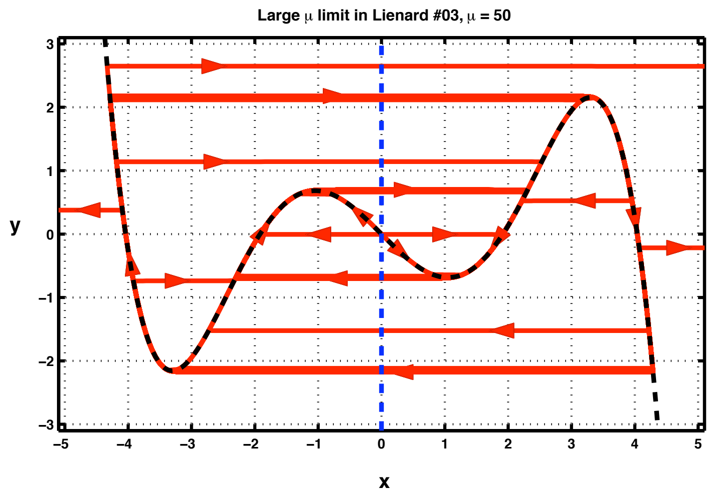
Using the facts above, it is now easy to figure out what the orbits do — see figure 1.1. The conclusion is that:

c1. There are **two limit cycles, both clockwise.** The larger one is **unstable**, and encloses the smaller one, which is **stable**.

The larger one consists (approximately) of two horizontal segments (from the branch I_0 to P_4, and from the branch I_4 to P_1), plus the complimentary pieces from the branches I_0 and I_4.

The smaller one consists (approximately) of two horizontal segments (from P_2 to the branch I_3, and from P_3 to the branch I_1), plus the complimentary pieces from the branches I_1 and I_3.

c2. The orbits that leave the larger limit cycle either zoom to infinity, or converge towards the smaller limit cycle. The orbits that leave the unstable node $\mathcal{C}$ converge towards the inner limit cycle.



Liénard system large μ limit #03. The dashed blue (resp. black) line indicates the zero of g (resp. the curve $y = f(x)$). The critical point $\mathcal{C}$ is an unstable node. The two limit cycles can be seen in thicker red than the other orbits.

Figure 1.1: Phase portrait for the system in (1.3 – 1.4).

Task #4 left to the reader. *Consider the solutions trapped near the branch I_0. Eventually these solutions leave the trapping region, either to the right or the left. There must be an orbit that is the transition between these two behaviors, and stays close to the branch all the way to infinity.* **Find an approximate expression for this orbit.**

Task #5 left to the reader. *There are two orbits that leave $\mathcal{C}$ (one up to the left, the other down to the right) and stay close to the branch I_2 all the way to either $\mathcal{P}2$ or $\mathcal{P}3$.* **Find an approximate expression for these orbits.**

2 Phase Plane Surgery #01

2.1 Statement: Phase Plane Surgery #01

Can a smooth vector field exist in the plane such that:

— The critical points are $P_1 = (-2, 0)$, $P_2 = (0, 0)$ and $P_3 = (2, 0)$.

— All the critical points are spirals.

— The circles with radii: $R_1 = 1$ centered at P_1, $R_2 = 4$ centered at P_2, and $R_3 = 1$ centered at P_3, are orbits.

Would your answer change if P_2 is a saddle?

In either case, if your answer is yes, sketch the way the orbits might look in an example satisfying the criteria above.

Challenge question: In either case, if your answer is yes, can you give an actual example (i.e.: write the vector field explicitly) that gives you a phase portrait with the same qualitative features (the closed orbits need not be circles for this).

2.2 Answer: Phase Plane Surgery #01

The answer to the first question in this problem is: NO.
The reason is that the circle of radius R_2 centered at P_2 would be an orbit enclosing all three critical points. Thus the sum of the indexes of the critical points would have to be one, which would contradict the fact that the critical points are all supposed to be spirals.

If, on the other hand, P_2 is a saddle, then a vector field with the given properties exists.
We will show this by actually constructing one. In our construction the closed orbits doing the job are not be circles, but this is a minor point: *The method used to construct the example can be adapted (at the price of much more algebra in the answer) to obtain closed orbits that are circles — or ellipses or any other closed curves one may desire.*

Let V be the potential defined by $\boldsymbol{V = -2\,x^2 + \frac{1}{4}\,x^4}$. For this potential
— $x = 0$ is a local maximum — with $V(0) = 0$.
— $x = \pm 2$ are local minimums — with $V(-2) = V(2) = -4$.
— Everywhere else $\dfrac{dV}{dx} \neq 0$.

Define $E = E(x,\,y)$ by
$$E = \frac{1}{2}\,y^2 + V(x). \tag{2.1}$$

Then

1. $\nabla E = (E_x,\,E_y) = \left(\dfrac{dV}{dx},\,y\right)$ vanishes at the points P_1, P_2, and P_3, **only.**
2. P_1 and P_3 are local minimums of the surface $z = E(x,\,y)$, while P_2 is a saddle.
3. The level curves $E = c =$ constant (with $-4 < c < 0$) have two components: one enclosing (only) P_1 and the other enclosing (only) P_3.
4. The level curves $E = c =$ constant (with $0 < c$) have only one component, which encloses all the P_j's and all the level curves with $-4 < E < 0$.

Consider now the phase plane system given by:

$$\begin{aligned}\frac{dx}{dt} &= -E_y - f(E)\,E_x,\\ \frac{dy}{dt} &= E_x - f(E)\,E_y,\end{aligned} \qquad \text{i.e.:} \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = (\nabla E)^{\perp} - f(E)\,\nabla E, \tag{2.2}$$

where $f = f(E)$ is some function to be defined later, and the superscript $\perp$ indicates rotation by $\pi/2$ of a vector in the plane.

Remark 2.1 (Motivation). *The flow provided by the system of equations in (2.2) has two components. The first one, given by $(\nabla E)^{\perp}$, induces a flow along the level lines of the surface $z = E(x,\,y)$ — for this component alone the orbits would be the level curves of E. The second component, given by $f\,\nabla E$, induces a flow normal to the level lines of E — for this component alone the orbits would correspond to the path of water flowing down/up*[1] *the gravity gradient on the surface $z = E$.*

Thus, for the system in (2.2), the level curves of E corresponding to zeros of $f(E)$ are orbits. On the other hand, when f is small, the orbits will (approximately) track the level curves of E — slowly drifting up or down the gradient of E, depending on the sign of f. Thus, given the properties of E — itemized below equation (2.1), it should be clear that: by judiciously choosing $f = f(E)$, we can obtain a system with the desired properties.

A **clarification regarding orbits and level lines:** *when above we say "the orbits would be level lines of E", or similar, it should be clear that this applies for* each *component of $E =$ constant, if $E =$ constant is made up by more than one curve — e.g.: when $-4 < E < 0$. The case $E = 0$ (saddle level) has three components, one where $x > 0$, another where $x < 0$, and the critical point P_2. Finally, the case $E = -4$ has two components: the critical points P_1 and P_3.* ♣

[1] Down if $f > 0$; up if $f < 0$.

Below we fill in the details of the intuitive arguments in the remark.

For the system in (2.2), we have
$$\left\{\begin{array}{rcl} f\,\dot{x}-\dot{y} & = & -(1+f^2)\,E_x, \\ f\,\dot{y}+\dot{x} & = & -(1+f^2)\,E_y, \end{array}\right. \tag{2.3}$$

from which it follows that **the critical points of (2.2) are exactly the P_j's** — since these are the only points where ∇E vanishes. It is also clear that, along orbits:

$$\frac{dE}{dt} = \dot{x}\,E_x + \dot{y}\,E_y = -\left(E_x^2 + E_y^2\right)\,f(E). \tag{2.4}$$

Hence: **Any level curve $E = E_0 =$ constant, such that $f(E_0) = 0$, is an orbit.** (2.5)

Now take
$$f = \epsilon\,(E^2 - 4), \tag{2.6}$$
where $\epsilon > 0$ is an arbitrary constant. With this choice for $f = f(E)$, note that:

5. The critical points $P_1 = (-2,\,0)$ and $P_3 = (2,\,0)$ are stable spirals if $\epsilon < \sqrt{2}/49 \approx 0.0673\ldots$, or stable improper nodes if $\epsilon = \sqrt{2}/49$, or stable nodes if $\sqrt{2}/49 < \epsilon$.

This follows from the 2×2 matrix of the linearized equations near the critical points, both given by:
$$A = \begin{bmatrix} -96\,\epsilon & -1 \\ 8 & -12\,\epsilon \end{bmatrix}.$$

6. The critical point $P_2 = (0,\,0)$ is always a saddle.

This follows from the 2×2 matrix of the linearized equations near the critical point, given by:
$$A = \begin{bmatrix} -16\,\epsilon & -1 \\ -4 & 4\,\epsilon \end{bmatrix}.$$

7. For **any orbit that is not a critical point,** (2.4) yields

$$\begin{array}{llll} \text{As} \quad t\to\infty & E\to 2 & \text{or} & E\to -4. \\ \text{As} \quad t\to -\infty & E\to -2 & \text{or} & E\to \infty. \end{array} \tag{2.7}$$

A sketch of a proof is at the end of this problem answer.

Proviso: If an orbit has $E\to\infty$ in the direction of increasing t, then the limit is achieved for a *finite* value of t. The other values ($E=\pm 2$ and $E=-4$) require either $t\to\infty$ or $t\to-\infty$.

Thus, **provided that $0 < \epsilon < \sqrt{2}/49 \approx 0.0673\ldots$, we have:**

8. The two level curves $E = -2$ are unstable limit cycles enclosing the critical points P_1 and P_3.

9. The level curve $E = 2$ is a stable limit cycle enclosing all the critical points.

10. The critical points P_1 and P_3 are stable spiral points, while P_2 is a saddle point.

Figure 2.1 shows the phase portrait of the system for the choice $\epsilon = 1/40$. The degree of stability of the limit cycles, and of the critical points P_1 and P_3, is controlled by ϵ — as should be obvious from (2.4). As ϵ grows, the limit cycles become more stable (resp. unstable), and at $\epsilon = \sqrt{2}/49$ the critical points P_1 and P_3 switch from spirals to nodes. Unless ϵ is fairly small, the orbits move away (towards) the limit cycles so fast that they appear as the spokes of bicycle tires near them.

It should be obvious that the shape and location of the limit cycles can be changed — by replacing E by some other function with different level curves. It is possible to get the limit cycles to have almost any desired shape (at the price of having to manufacture a complicated [2] function E).

[2] An explicit expression for E might not even be possible. But the surface $z = E(x,\,y)$ need not be "geometrically" more complicated than the one given by (2.1).

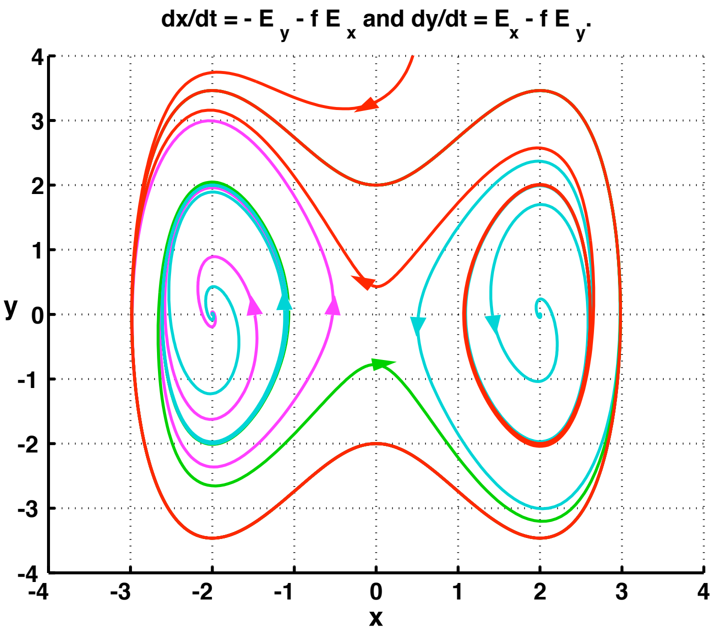


Figure 2.1: Phase portrait for the system in (2.2), with $\epsilon = 0.025$ in (2.6). The two unstable limit cycles are given by the components of the level curve $E = -2$, and the single stable limit cycle is given by the level curve $E = 2$.

Sketch of the proof of (2.7). *Consider an orbit that is not a critical point. Then* $E_x^2 + E_y^2 > 0$ *everywhere on the orbit — although* $E_x^2 + E_y^2$ *may approach zero if the orbit approaches a critical point as either* $t \to \infty$*, or* $t \to -\infty$*. Then (2.4) shows that* E *is increasing for* $-2 < E < 2$*, and decreasing for* $|E| > 2$. Note that E is restricted to the range $-4 \leq E < \infty$.

3 Simple Poincaré Map for a limit cycle #02

3.1 Statement: Simple Poincaré Map for a limit cycle #02

Consider the following autonomous phase plane system

$$\left.\begin{aligned} \frac{dx}{dt} &= (x^2 + y^4)\left(\nu\, x - \tfrac{\nu}{4}\, x^3 - x^2\, y - \nu\, x\, y^2 - 4\, y^3\right), \\ \frac{dy}{dt} &= (x^2 + y^4)\left(\nu\, y + \tfrac{1}{4}\, x^3 - \tfrac{\nu}{4}\, x^2\, y + x\, y^2 - \nu\, y^3\right), \end{aligned}\right\} \quad \text{where } \nu > 0. \tag{3.1}$$

This system has a periodic solution **(show this)**, which can be written in the form

$$x = 2\cos\Phi, \quad y = \sin\Phi, \quad \text{where} \quad \frac{d\Phi}{dt} = 2\,(x^2 + y^4) = 2\,(1 + \cos^2\Phi)^2. \tag{3.2}$$

This solution produces an orbit going through the point $x = 0$, $y = 1$ in the phase plane. The orbit is an ellipse, as (3.2) shows.[3]

Construct (either numerically[4] or analytically) a Poincaré map near this orbit, and use it to show that the orbit is a stable limit cycle. *Define the Poincaré map* $\boldsymbol{z} \to \boldsymbol{u} = \boldsymbol{P}(\boldsymbol{z})$ as follows:

[3] Note that $\boldsymbol{\Phi}$ is a strictly increasing function of time.

[4] If you do it numerically, keep ν as a variable and check your answers for several values — say: $\nu = 0.1,\ 0.5,\ 1,\ 2,\ 5$.

- For every sufficiently small $\boldsymbol{z}$, let $\boldsymbol{x = X(t,\, z)}$ and $\boldsymbol{y = Y(t,\, z)}$ be the solution of (3.1) defined by $\boldsymbol{X(0,\, z) = 0}$ and $\boldsymbol{Y(0,\, z) = 1 + z}$.
- For this solution the polar angle $\boldsymbol{\theta}$ in the phase plane is an *increasing function of time*, starting at $\boldsymbol{\theta = \frac{1}{2}\pi}$ for $\boldsymbol{t = 0}$. Thus, there is a time $\boldsymbol{t = t_z}$ at which the solution reaches $\boldsymbol{\theta = \frac{5}{2}\pi}$ (note that $\boldsymbol{t_z}$ is a function of $\boldsymbol{z}$). Then take $\boldsymbol{u = Y(t_z,\, z) - 1}$.

Hint. Because t_z is a function of z, unknown a priori, the definition of the Poincaré map above is a bit awkward to implement. To avoid having to calculate t_z for each solution, it is a good idea to use a parameter other than time to describe the orbits. For example, if the equations are written in terms of a parameter such as the polar angle — namely $\boldsymbol{\frac{dx}{d\theta} = F(x,\, y)}$ and $\boldsymbol{\frac{dy}{d\theta} = G(x,\, y)}$, then the Poincaré map is easier to describe, as θ varies from $\theta = \frac{1}{2}\pi$ to $\theta = \frac{5}{2}\pi$ in every one of the orbits needed to compute $u = P(z)$. Note that this is just a "for example", using the polar angle is not the best choice. *Scale the variables first, so that the limit circle is a circle, not an ellipse.* ♣

Small challenge: You should be able to write $\boldsymbol{P}$ analytically. The formula is not even messy.

3.2 Answer: Simple Poincaré Map for a limit cycle #02

Since the questions of interest here have to do with the topology of the orbits in phase space, it does not matter if we parameterize the orbits with time, or some other parameter, as long as they both yield the same direction of travel along the orbits (so stability is not affected). Thus, introduce a **"new" time along the orbits**, as follows:

$$\frac{d\tau}{dt} = x^2 + y^4, \quad \Longrightarrow \quad \left.\begin{aligned} \frac{dx}{d\tau} &= \nu\, x - \tfrac{\nu}{4}\, x^3 - x^2\, y - \nu\, x\, y^2 - 4\, y^3, \\ \frac{dy}{d\tau} &= \nu\, y + \tfrac{1}{4}\, x^3 - \tfrac{\nu}{4}\, x^2\, y + x\, y^2 - \nu\, y^3. \end{aligned}\right\} \tag{3.3}$$

Except for the critical point at the origin, $d\tau/dt > 0$, so that τ is acceptable as an orbit parameter. Now we notice that we can write the equations in the form

$$\left.\begin{aligned} \frac{dx}{d\tau} &= \nu\, x\, (1 - \tfrac{1}{4}\, x^2 - y^2) - 4\, y\, (\tfrac{1}{4}\, x^2 + y^2), \\ \frac{dy}{d\tau} &= \nu\, y\, (1 - \tfrac{1}{4}\, x^2 - y^2) + \, x\, (\tfrac{1}{4}\, x^2 + y^2). \end{aligned}\right\} \tag{3.4}$$

The combination $\dfrac{1}{4}\, x^2 + y^2$ in these equations suggests that we introduce elliptic-polar coordinates

$$x = 2\, R\, \cos\Phi \quad \text{and} \quad y = R\, \sin\Phi \qquad \Longleftrightarrow \qquad R^2 = \frac{1}{4}\, x^2 + y^2 \quad \text{and} \quad \tan\Phi = 2\, \frac{y}{x}. \tag{3.5}$$

Hence[5] $\dfrac{dR}{d\tau} = \nu\, R\, (1 - R^2)$ and $\dfrac{d\Phi}{d\tau} = 2\, R^2$, so that

$$\frac{dR^2}{d\Phi} = \nu\, \left(1 - R^2\right) \quad \Longrightarrow \quad R^2 = 1 + ((R_0)^2 - 1)\, \exp\left(\nu\, \left(\tfrac{\pi}{2} - \Phi\right)\right), \tag{3.6}$$

where R_0 is the value of R for $\Phi = \pi/2$. **The Poincaré map now follows** from this last equation. We want to find the value $\boldsymbol{R = 1 + u}$, at $\boldsymbol{\Phi = 5\,\pi/2}$, of the orbit for which $\boldsymbol{R_0 = 1 + z}$. Hence

$$\boldsymbol{(1 + u)^2 = 1 + (z^2 + 2\, z)\, e^{-2\,\pi\,\nu}} \quad \Longrightarrow \quad \boldsymbol{u = P(z) = -1 + \sqrt{1 + (z^2 + 2\, z)\, e^{-2\,\pi\,\nu}}.} \tag{3.7}$$

In particular: $\boxed{\boldsymbol{P(0) = 0 \text{ and } 0 < \frac{dP}{dz}(0) = e^{-2\,\pi\,\nu} < 1.}}$ $\left\{\begin{array}{l}\text{Thus the limit cycle} \\ \text{in (3.2) is stable.}\end{array}\right.$

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[5] Note that, for $\boldsymbol{R = 1}$, these equations provide the periodic solution in equation (3.2).

4 Problem 07.02.x1 (area evolution)

4.1 Statement for problem 07.02.x1

Consider some arbitrary orbit, Γ, for the phase plane system

$$\frac{d\vec{r}}{dt} = \vec{F}(\vec{r}) \quad \text{where} \quad \vec{r} = (x,\, y)^T, \quad \vec{F} = (f(x,\, y),\, g(x,\, y))^T, \tag{4.1}$$

and $\vec{F}$ has continuous partial derivatives up to (at least) second order. That is: Γ is the curve in the plane given by some solution $\vec{r} = \vec{r}_\gamma(t)$ to (4.1). Then

A. Let $\Omega = \Omega(t)$ be an "infinitesimal" region that is being advected, along Γ, by the flow given by (4.1). For example:

- **A1.** Let $\Omega(0)$ be a disk of "infinitesimal" radius dr, centered at $\vec{r}_\gamma(0)$.
- **A2.** For every point $\vec{r}_p^{\,0} \in \Omega(0)$, let $\vec{r} = \vec{r}_p(t)$ be the solution to (4.1) defined by the initial data $\vec{r}_p(0) = \vec{r}_p^{\,0}$.
- **A3.** Then, at any time t_*, the set $\Omega(t_*)$ is given by all the points $\vec{r}_p(t_*)$, where $\vec{r}_p^{\,0}$ runs over all the points in $\Omega(0)$.

Note that $\Omega(0)$ need not be a disk. Any infinitesimal region containing $\vec{r}_\gamma(0)$ will do. All we need is that the notion of area applies to it — see item **B**.

B. Let $\mathcal{A} = \mathcal{A}(t)$ be the area of $\Omega(t)$.

Find a differential equation for the time evolution of $\mathcal{A}$. *The equation that you will find is trivially extended to higher dimensions — e.g. to characterize the evolution of the volume in a 3-D phase space.*

Hints.

h1. First, introduce the vector $\delta\vec{r} = \delta\vec{r}(t) = \vec{r}_p - \vec{r}_\gamma$ for every point in $\Omega(t)$. This vector characterizes the evolution of the "shape" of Ω as the set moves along Γ. In order to calculate how $\mathcal{A}(t)$ evolves, you only need to know how the $\delta\vec{r}$ vectors evolve.

h2. For every vector $\delta\vec{r}$, write an equation giving $\delta\vec{r}(t + dt)$ in terms of $\delta\vec{r}(t)$ and the partial derivatives of $\vec{F}$ along Γ. Since you are dealing with infinitesimal terms, you can neglect higher order terms, so as to obtain a relationship from $\delta\vec{r}(t)$ to $\delta\vec{r}(t + dt)$ given by a linear transformation. Make sure that this linear transformation correctly includes the $O(dt)$ terms, which you will need to calculate time derivatives.

h3. From the transformation in item **h2** derive a relationship between $\mathcal{A}(t + dt)$ and $\mathcal{A}(t)$ — use the fact that, for linear transformations, areas are related by the absolute value of the determinant. You need to calculate the determinant only up to $O(dt)$.

h4. Use the result in item **h3** to calculate the time derivative of $\mathcal{A}$, and obtain the differential equation.

4.2 Answer for problem 07.02.x1

At any time t, we can write

$$\vec{r}_\gamma(t + dt) = \vec{r}_\gamma(t) + \vec{F}_\gamma(t)\, dt, \tag{4.2}$$
$$\vec{r}_p(t + dt) = \vec{r}_p(t) + \vec{F}_p(t)\, dt, \tag{4.3}$$

where we have neglected $O\left((dt)^2\right)$ contributions, $\vec{F}_\gamma = F(\vec{r}_\gamma)$, $\vec{F}_p = F(\vec{r}_p)$, and $\vec{r}_p = \vec{r}_p(t)$ tracks an arbitrary point in $\Omega(t)$ — as in item A2. Hence we can write

$$\begin{aligned} \delta\vec{r}\,(t + dt) &= \delta\vec{r}\,(t) + \left(\vec{F}_p(t) - \vec{F}_\gamma(t)\right)\, dt \\ &= \left(I + M_\gamma(t)\, dt\right)\, \delta\vec{r}\,(t), \end{aligned} \tag{4.4}$$

where $\delta\vec{r} = \vec{r}_p - \vec{r}_\gamma$, I is the identity matrix, M is the matrix of partial derivatives of $\vec{F}$

$$M = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix},$$

$M_\gamma = M(\vec{r}_\gamma)$, and we have neglected $O\left((dr)^2\, dt\right)$ terms to arrive at the second line in (4.4). It follows that

$$\begin{aligned} \mathcal{A}(t+dt) &= \det\left(I + M_\gamma(t)\, dt\right) \mathcal{A}(t) \\ &= \left(1 + \text{Trace}\left(M_\gamma(t)\right)\, dt\right) \mathcal{A}(t), \end{aligned} \tag{4.5}$$

where (again) we have neglected $O\left((dt)^2\right)$ terms when computing the determinant. From this last equation we obtain

$$\boldsymbol{\frac{d}{dt}\, \mathcal{A} = \textbf{div}\left(\vec{F}\right)\, \mathcal{A},} \tag{4.6}$$

where we have used that Trace $\left(M_\gamma(t)\right) = \text{div}\left(\vec{F}\right)$, **with the divergence evaluated along Γ.**

A second, more "mathematical", derivation follows:

Define $\boldsymbol{\vec{R} = \vec{R}(t,\, \vec{r}\,)}$ by

$$\frac{\partial}{\partial t}\,\vec{R} = \vec{F}\left(\vec{R}\right) \quad \text{with} \quad \vec{R}\,(0,\, \vec{r}\,) = \vec{r}. \tag{4.7}$$

Thus $\vec{R}$ characterizes the whole flow given by the system in (4.1).
In particular, $\boldsymbol{\vec{r}_\gamma(t) = \vec{R}\,(t,\, \vec{r}_\gamma(0))}$ and $\boldsymbol{\vec{r}_p(t) = \vec{R}\left(t,\, \vec{r}_p^{\,0}\right)}$ for every $\boldsymbol{\vec{r}_p^{\,0} \in \Omega(0)}$.

Let now $\boldsymbol{S = S(t,\, \vec{r}\,)}$ be the matrix of the partial derivatives of $\vec{R}$ with respect to $\vec{r}$, with $\boldsymbol{S_\gamma = S(t,\, \vec{r}_\gamma(0))}$. Then

$$\mathcal{A}(t) = J(t)\, \mathcal{A}(0), \quad \text{where} \quad J = \det(S_\gamma) \tag{4.8}$$

is the Jacobian — along Γ — of the transformation from $\Omega(0)$ to $\Omega(t)$ defined by $\vec{R}$. Now we have

1. $\boldsymbol{\frac{\partial}{\partial t}\, S = M\, S}$, where M is the matrix of partial derivatives of $\vec{F}$.
   This follows by taking partial derivatives, with respect to $\vec{r}$, of the equation for $\vec{R}$ in (4.7).
2. $\boldsymbol{\ln(J) = \ln\left(\det(S_\gamma)\right) = \textbf{Trace}\left(\ln(S_\gamma)\right)}$ .... Why do we need this? See remark **4.2**.
3. $\boldsymbol{\frac{1}{J}\,\frac{dJ}{dt} = \textbf{Trace}\left(\frac{dS_\gamma}{dt}\, S_\gamma^{-1}\right) = \textbf{Trace}\left(M_\gamma\right)}$.
   This follows by taking the time derivative of the equation in item **2**, and then using the result in item **1**.

It should be now clear that, from (4.8) and the result in item **3**, the same equation (4.6) obtained earlier for the time evolution of $\mathcal{A}$ follows.

Remark 4.2 *In the calculation above we need to calculate the time derivative of J, which is defined as the determinant of a matrix. Directly calculating the derivatives of a determinant is, generally, rather messy. On the other hand, the formula in item* **2** *transforms this into the calculation of the derivative of the trace of the logarithm of a matrix. While the derivative of the logarithm of a matrix is, itself, hard to compute (because matrix multiplication is not commutative), the trace fixes this problem — since Trace(AB) = Trace(BA) for any square matrices A and B — and one can then take the derivative as if scalars, instead of matrices, were involved.*

5 Problem 07.05.06 - Strogatz (Biased van der Pol)

5.1 Statement for problem 07.05.06

Suppose the van der Pol oscillator is biased by a constant force:

$$\frac{d^2x}{dt^2} + \mu\,(x^2-1)\,\frac{dx}{dt} + x = a, \tag{5.1}$$

where a can be positive, negative, or zero. (Assume $\mu > 0$ as usual).

a) Find and classify all the fixed points.

b) Plot the nullclines in the Liénard plane. Show that if they intersect on the *middle* branch of the cubic nullcline, the corresponding fixed point is unstable.

c) For $\mu \gg 1$, show that the system has a stable limit cycle if and only if $|a| < a_c$, where a_c is to be determined. (*Hint. Use the Liénard plane.*)

d) Sketch the phase portrait for a slightly greater than a_c. Show that the system is *excitable* — it has a globally attracting fixed point, but some (small, but not infinitesimal) disturbances can send the system on a long excursion through phase space before returning to the fixed point; compare with Exercise 4.5.3.

This system is closely related to the Fitzhugh-Nagumo model of neural activity; for an introduction see Murray, J. (1989) *Mathematical Biology* (Springer, New York) or Edelstein-Keshet, L. (1988) *Mathematical Models in Biology* (Random House, New York).

5.2 Answer for problem 07.05.06

We begin by writing the *Liénard plane* form of the equations. The system in (5.1) can be written in the form:

$$\frac{d}{dt}\left(\frac{dx}{dt} + \mu(\frac{1}{3}x^3 - x)\right) + x = a.$$

Thus, in terms of

$$y = \frac{1}{\mu}\left(\frac{dx}{dt} + \mu(\frac{1}{3}x^3 - x)\right),$$

we can write the equations in the *Liénard plane* as:

$$\frac{dx}{dt} = \mu\left(y - \frac{1}{3}x^3 + x\right) \qquad \text{and} \qquad \frac{dy}{dt} = \frac{1}{\mu}(a-x). \tag{5.2}$$

The **equations are invariant under the transformation $x \to -x$, $y \to -y$, and $a \to -a$.** Thus:

$$\textbf{There is no loss of generality in assuming that } a > 0. \tag{5.3}$$

a) Fixed points.
The system in (5.2) has only one fixed point, namely $P = (x_0,\, y_0) = (a,\, \frac{1}{3}a^3 - a)$. The linearized system near this critical point is:

$$\frac{dX}{dt} = \mu(1-a^2)X + \mu Y \qquad \text{and} \qquad \frac{dY}{dt} = -\frac{1}{\mu}X, \tag{5.4}$$

with eigenvalues

$$\lambda = \frac{1}{2}\left(\mu(1-a^2) \pm \sqrt{\mu^2(1-a^2)^2 - 4}\right). \tag{5.5}$$

Thus:

a1)	For	$1+\frac{2}{\mu}$	$<$	a^2			P is a **stable node.**	Figure 5.1-A.
a2)	For	1	$<$	a^2	$<$	$1+\frac{2}{\mu}$	P is a **stable spiral.**	Figure 5.1-B.
a3)	For	$1-\frac{2}{\mu}$	$<$	a^2	$<$	1	P is an **unstable spiral.**	Figure 5.2-A.
a4)	For			a^2	$<$	$1-\frac{2}{\mu}$	P is an **unstable node.**	Figure 5.2-B.

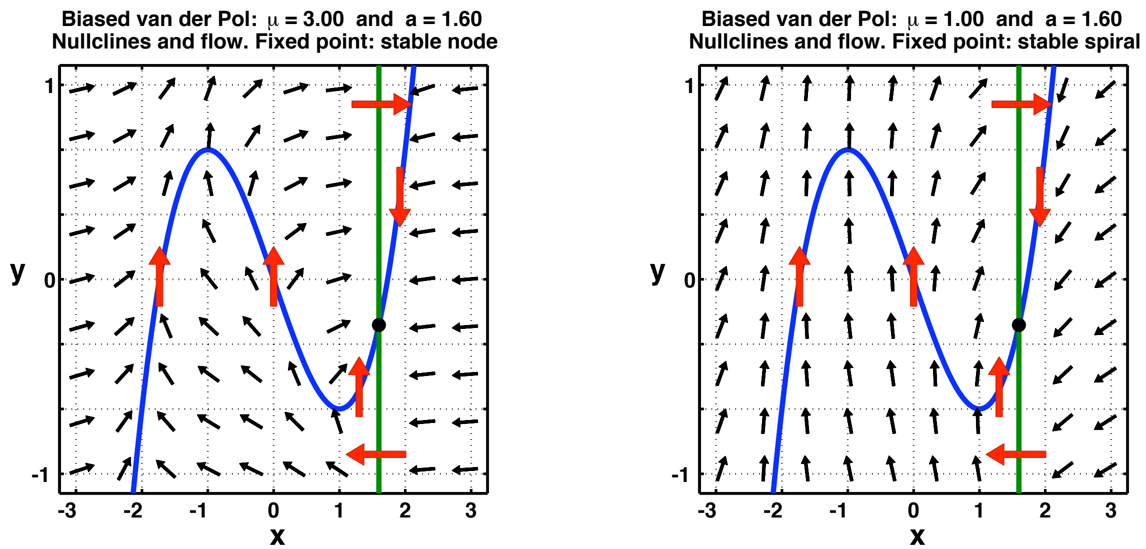


Figure 5.1: (Problem 07.05.06). Biased van der Pol; equation (5.2). Nullclines and flow field.
A (left). Fixed point: stable node, $1+\frac{2}{\mu}<a^2$. **B (right).** Fixed point: stable spiral, $1<a^2<1+\frac{2}{\mu}$.

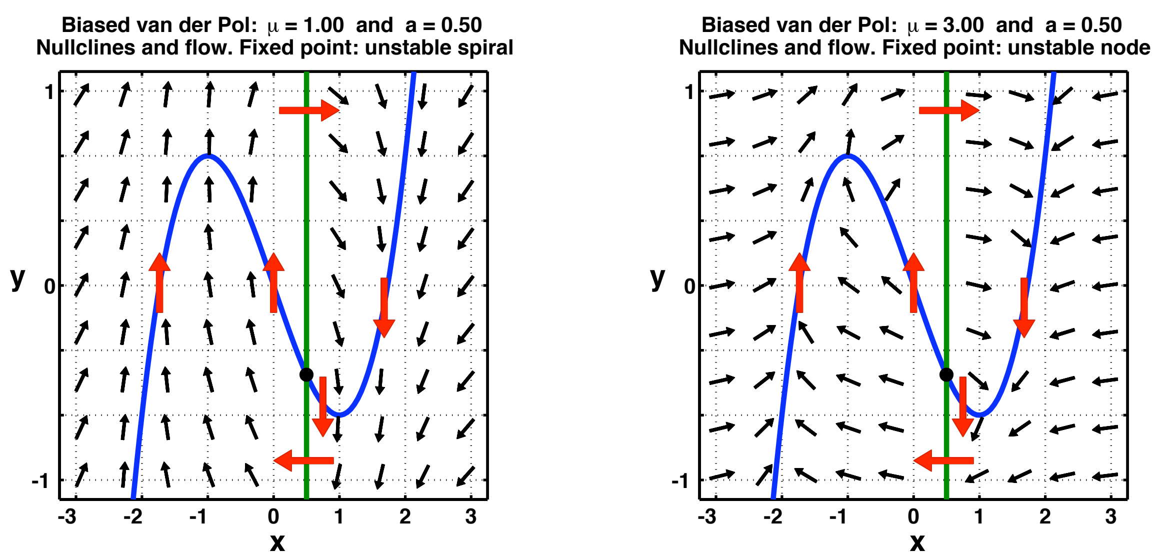


Figure 5.2: (Problem 07.05.06). Biased van der Pol; equation (5.2). Nullclines and flow field.
A (left). Fixed point: unstable spiral, $1-\frac{2}{\mu}<a^2<1$. **B (right).** Fixed point: unstable node, $a^2<1-\frac{2}{\mu}$.

The **borderline values** $a^2=1+2/\mu$, $a^2=1$, and $a^2=1-2/\mu$ correspond (in the linearized regime) to a stable improper node (double eigenvalue with geometric multiplicity one), a center and an unstable improper node, respectively. In order to ascertain the detailed behavior of the orbits near the critical point in these cases, a **nonlinear analysis is required.** However, the *only situation* we actually *have to worry about is the one that occurs for* $a = 1$ *and the critical point is a linear center* — because this is the only case where the nonlinear terms decide the stability properties of the fixed point. In the other cases the nonlinear terms may turn the improper node into a weak spiral, but they cannot change the stability properties of the critical point. In what follows we will do a **nonlinear analysis near the center,** for the case $a = 1$. We will use a **method that is an alternative to the two-timing asymptotic techniques introduced in the lectures.**

Nonlinear analysis near the linear center.

Assume $\boldsymbol{a = 1}$, and write $x = 1 + X$ and $y = -\frac{2}{3} + \frac{1}{\mu} Y$. The equations are then

$$\frac{dX}{dt} = Y - \mu\left(X^2 + \frac{1}{3}X^3\right) \qquad \text{and} \qquad \frac{dY}{dt} = -X. \tag{5.6}$$

As expected, at leading order for $X^2 + Y^2 \ll 1$, these are the harmonic oscillator equations. Consider now the harmonic oscillator energy: $\boldsymbol{E = \frac{1}{2}\left(X^2 + Y^2\right)}$. This quantity is conserved by the linear part of the equations above in (5.6), but the nonlinear terms change the conservation equation $\dot{E} = 0$ to:

$$\frac{dE}{dt} = -\mu\left(X^3 + \frac{1}{3}X^4\right). \tag{5.7}$$

We claim now that it is possible to *correct* E, by higher order *cubic* terms in X and Y, in such a way that the leading order *cubic* terms on the right hand side in (5.7) are eliminated. To do this we must find a combination of terms of the form $X^n Y^{3-n}$ (with $n = 0, 1, 2, 3$), whose time derivative — as given by equation (5.6) — is, to leading order, equal to μX^3. Adding then this combination to E will produce the desired result. This turns out to be not too difficult a task: a few simple manipulations using equation (5.6) shows that the following four equalities apply:

$$\left.\begin{array}{rcl} \dfrac{d}{dt}\left(-X^2Y - \dfrac{2}{3}Y^3\right) & = & X^3 \\ \dfrac{d}{dt}\left(\dfrac{1}{3}X^3\right) & = & X^2Y \\ \dfrac{d}{dt}\left(-\dfrac{1}{3}Y^3\right) & = & XY^2 \\ \dfrac{d}{dt}\left(XY^2 + \dfrac{2}{3}X^3\right) & = & Y^3 \end{array}\right\} + O\left((X^2+Y^2)^2\right). \tag{5.8}$$

Therefore any possible combination of cubic terms on the right in (5.7) can be eliminated. Thus consider the **corrected energy**

$$E_c = \frac{1}{2}\left(X^2 + Y^2\right) - \mu\left(X^2Y + \frac{2}{3}Y^3\right), \tag{5.9}$$

which satisfies:

$$\frac{dE_c}{dt} = -\frac{1}{3}\mu X^4 + 2\mu^2 X^3 Y + \frac{2}{3}\mu^2 X^4 Y. \tag{5.10}$$

Consider now an arbitrary solution *near* the center (say, within a distance $0 < \epsilon \ll 1$ of the center). For any *finite* period of time, we know that we can write

$$X = \epsilon\cos(t - t_0) + O(\epsilon^2) \qquad \text{and} \qquad Y = -\epsilon\sin(t - t_0) + O(\epsilon^2), \tag{5.11}$$

where t_0 is some constant. Substituting this into the right hand side in (5.10), and integrating from $t = 0$ to $t = 2\pi$, we can compute the leading order change in E_c as the solution goes once around the center. This yields:

$$\Delta E_c = -\frac{\mu}{8}\epsilon^4 + O(\epsilon^5). \tag{5.12}$$

Thus E decreases each time the solution goes around the fixed point. Since the surface $z = E(x,\, y)$ has a local minimum at the critical point, we conclude that:

$$\textbf{For } a = 1 \textbf{ the critical point is a nonlinear spiral.} \tag{5.13}$$

Remark 5.3 *As we pointed out earlier, the approach used above to arrive at the conclusion in (5.13) is* ***something that one can always do near a critical point that is a center, no matter what the nonlinearity.*** *This follows because the equations near a center can always be manipulated into a form such that the linear part looks as in (5.6). Thus E will satisfy an equation similar to (5.7), with some nonlinearity on the right hand side. Then (5.8) shows that E can be modified to some E_c (by adding cubic terms to E) so that the leading order time derivative of E_c is quartic, as in (5.10). Then the argument above in (5.11 – 5.12) will — generally — give a nonzero value to ΔE_c, meaning that we will know if the point becomes a stable or an unstable spiral point due to the nonlinear effects.*

The approach will fail when the calculation yields $\Delta E_c = O(\epsilon^5)$ above in (5.12). In this case one must add further corrections[6] *to E_c, so as to compute ΔE_c with higher accuracy. This will require the elimination of both the fourth and fifth order nonlinearities on the right hand side in (5.10). We further note that this failure will occur, precisely, under the same circumstances where the alternative two-timing approach will require computation of the asymptotic approximation to an order higher than the normal cubic.*

The ***advantage*** *of the approach here over the two-timing asymptotic approximation introduced in the lectures is that it requires quite a bit* ***less computation.****. The* ***disadvantage*** *is that it gives* ***less detailed information*** *— something that, many times is not too important. It is also somewhat less general and in problems involving many dimensions it is not necessarily as simple to implement as in 2-D. At any rate:* ***it is worth keeping it in mind whenever one has to elucidate stability issues in situations where linearized theory is not good enough.***

Another "advantage" of this approach, is that it is often easy to recast the argument in a form that is mathematically rigorous — something that is much harder, or impossible, with two-timing calculations. For example, in our case here, the argument from (5.10) to (5.12) is not rigorous, because it involves the more-or-less hand-waving step (5.11). However, note that, of the two quartic terms on the right in (5.10), one is a perfect derivative at leading order. Thus we can write:

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left(E_c - \frac{1}{2}\mu^2 X^4\right) = -\frac{1}{3}\mu X^4 + O\left((X^2+Y^2)^{5/4}\right).$$

But this means that, in a small enough neighborhood of the critical point, $\mathcal{E}$ is decreasing. This shows that the point must be a stable spiral — without any need to invoke (5.11).

b) Nullclines.

Figures 5.1 and 5.2 show plots of the nullclines (and the flow field) for the four typical cases (as outlined in **(a1)** through **(a4)** at the beginning of part **(a)**) that can arise for $a > 0$.

It is clear that instability of the critical point (figure 5.2) corresponds to the nullclines intersecting on the *middle* branch of the cubic nullcline.

c) Case $\mu \gg 1$.

When $\mu \gg 1$, the flow is nearly *horizontal*[7] *to the right,* for $y > (1/3)\,x^3 - x$; and nearly *horizontal to the left,* for $y < (1/3)\,x^3 - x$. It is only in a neighborhood of size $O(1/\mu)$ near the nullcline $y = (1/3)\,x^3 - x$ that significant motion in the vertical direction takes place. In this neighborhood the *orbits move down for $x > a$ and up for $x < a$.* **Two distinct situations can then arise:**

[6] This is possible: the vanishing of ΔE_c means that the $O(\epsilon^4)$ terms on the right hand side in (5.10) are exact derivatives.

[7] Parallel to the x-axis.

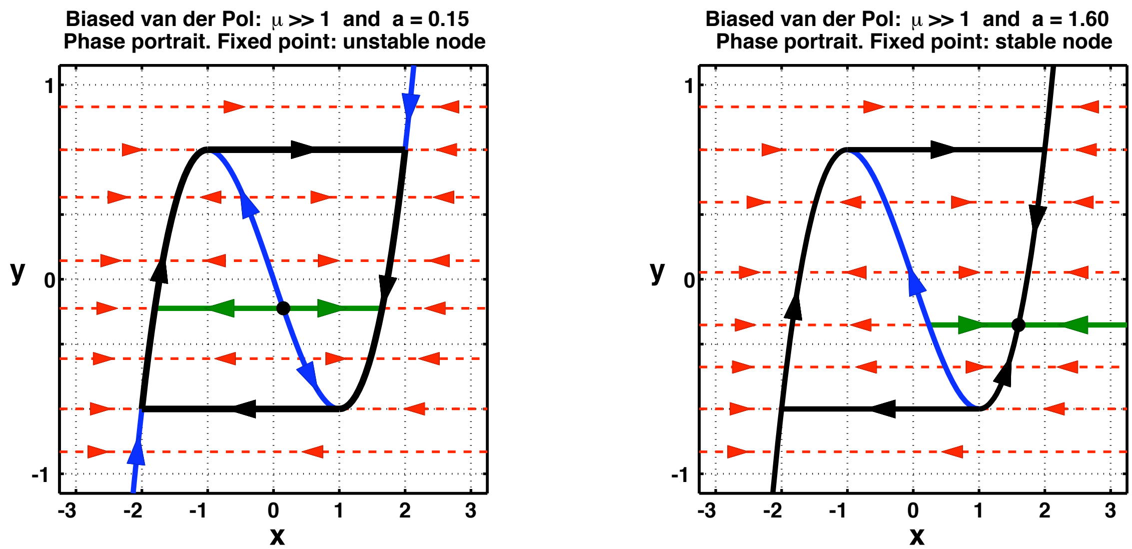


Figure 5.3: (Problem 07.05.06). Biased van der Pol. Phase portrait for equation (5.2) with $\mu \gg 1$.
A (left). Case with a limit cycle: $a^2 < 1$. **B (right).** Case without a limit cycle: $1 < a^2$.

c1) Limit cycle: $a < a_c = 1$.

A typical phase portrait for this case is shown in figure 5.3-A. Because the dividing point[8] between y increasing and y decreasing (near the nullcline $y = \frac{1}{3}x^3 - x$) is in the middle branch of the cubic; a situation arises where **all the orbits approach a limit cycle.** The argument is exactly the same we used for the relaxation oscillations in the un-biased van der Pol equation.

Figure 5.3-A shows some typical orbits, the limit cycle, the nullcline $y = \frac{1}{3}x^3 - x$, and the principal curves for the node at the critical point. Note that the middle branch of the nullcline approximates the principal curve along which all orbits but two leave the node.

You should think carefully about what is going on with the orbits as they approach the nullcline. They *do not merge with it*, but stay so close than in the figure they end up looking as the same curve. In particular, *sketch a blown up picture of the local orbit geometry near the points* $(x, y) = \pm(1, -2/3)$ (the local maximum and minimum of the nullcline), and along the middle branch of the nullcline.

c2) Global attracting fixed point $a > a_c = 1$.

A typical phase portrait for this case is shown in figure 5.3-B. Now the dividing point between y increasing and y decreasing is **not** in the middle branch of the cubic. It is then clear that this leads to the situation shown in figure 5.3-B, **without a limit cycle, and with all solutions approaching the critical point as $t \to \infty$.**

Again: think carefully about the behavior of the orbits near the nullcline; in particular of the local orbit geometry near $(x, y) = \pm(1, -2/3)$ and the middle branch of the nullcline.

A final note: figure 5.3-B corresponds to the case $a < 2$. For $a > 2$ the critical point moves beyond the S shaped part of the nullcline and some minor (and obvious) changes are needed in the phase portrait.

An interesting question arises now:

What happens near the critical threshold, i.e.: $a \approx a_c = 1$?

From **(a2)** and **(a3)** we see that[9] as a decreases through $a = 1$, the critical point switches from a stable spiral to an unstable spiral. On the other hand, (5.13) shows that the nonlinearity is stabilizing near the critical

[8] That is: the critical point for the equations.
[9] For $\mu > 0$ fixed.

point. Thus:

A supercritical Hopf bifurcation occurs at $a = 1$ ($\mu > 0$ fixed).

This fact creates a slight puzzle when viewed in the light of the the result above in **(c1)** and the phase portrait in figure 5.3-A. Namely: for a slightly less than 1, the limit cycle is supposed to have size $O(\sqrt{1-a})$, while figure 5.3-A indicates a limit cycle of size $O(1)$. The puzzle is resolved by realizing that the situations considered in **(c1)** and **(c2)** correspond to a fixed and $\mu \to \infty$. Then:

c3) This explains why the critical point is a node in figure 5.3-A. In the situation considered in **(c1)** the critical point is always a node, as follows from **(a4)** for $a < 1$ fixed and $\mu \to \infty$.

c4) The Hopf bifurcation theory assures us that, for $a < 1$ "close" to 1, the limit cycle will have size $O(\sqrt{1-a})$. But it says nothing about how close is close. In this particular case with the large parameter μ in the equation, close means **very close**. Clearly, much closer that $O(1/\mu)$, since the Hopf bifurcation theory assumes a situation where the departure of the critical point from a center is small — if $1 - a = O(1/\mu)$, (5.5) shows that the departures will be anything but small. Thus, in the regime considered in **(c1)** and in figure 5.3-A, the limit cycle need not be small at all.

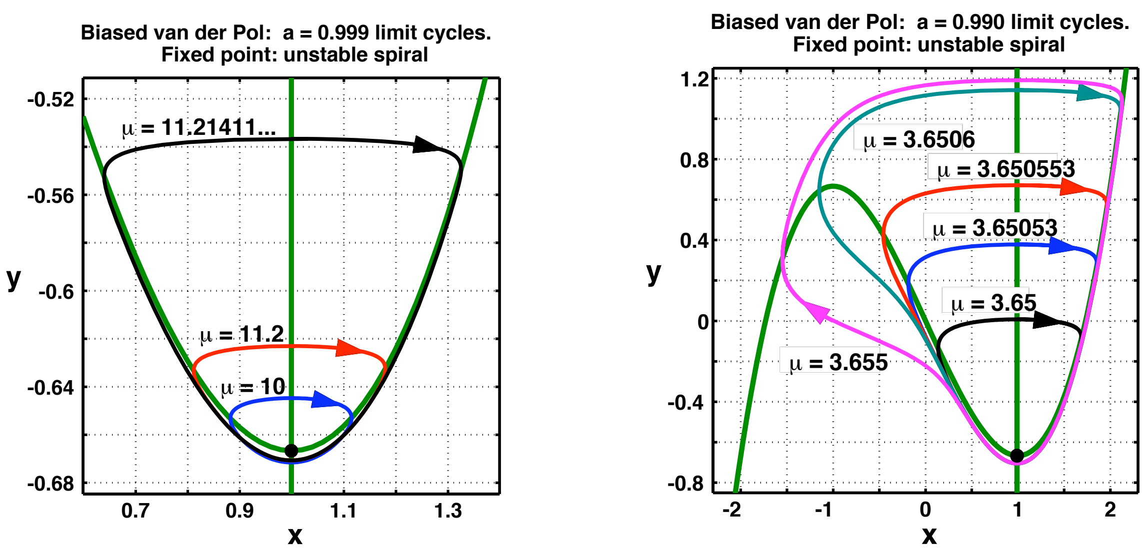


Figure 5.4: (Problem 07.05.06). Biased van der Pol; equation (5.2) for $0 < 1 - a \ll 1$ fixed. The limit cycles grow in size as μ grows. **A (left).** Case $a = 0.999$. **B (right).** Case $a = 0.990$.

The behavior of the limit cycle for $0 < 1 - a \ll 1$ fixed, as μ changes, is illustrated in figure 5.4 — where the nullclines and limit cycles for various values of μ are plotted. When $1 - a \ll 1/\mu$ the limit cycle is in the regime where the asymptotic Hopf bifurcation theory applies, but it is very small.[10] As μ grows, the limit cycle also grows in size. The cycle grows at first in such a way that it approximates the right dip in the $\dot{x} = 0$ nullcline, with the loop completed by a nearly horizontal jump from the middle unstable branch to the right stable branch of the nullcline. Once it reaches the maximum size it can achieve in this fashion, it continues growing by having a jump across to the left (instead of the right) stable branch from the middle branch. This switch is done when the jumping place to the right reaches the top of the left "mountain". At first the jump to the left is small — cutting across the top. But, as μ continues to grow, it travels down. Eventually the limit cycle ends up enclosing the whole S shaped part of the nullcline. Then a situation like the one described in figure 5.3 is reached.

[10] For μ large it is smaller than the $O(1/\mu)$ region near the nullcline — which in the figures it is approximated by a just line.

The behavior described in the prior paragraph is true in only a very rough sense when $(1-a)$ is moderately small. However, when $(1-a)$ is very small, the limit cycle reaches a finite size only for μ fairly large. Then the description in the paragraph above is quite accurate. Computation of the limit cycle as it follows the described behavior is very hard, for two reasons:

A. The jumps across from the middle branch to the side branches of the nullcline are very sensitive to numerical errors, since the middle branch is extremely unstable for $\mu \gg 1$. Thus, very small errors can dramatically alter the jumping place — thus the overall result of the calculation.

B. The limit cycle shape is very sensitive to the value of μ (this is, actually, a consequence of **A.**) This can be seen in figure 5.3-A, where very small changes in μ cause large changes in the limit cycle size.

It is because of this difficulty that the limit cycles shown in figure 5.3 are for regimes where μ is not very large. Thus the jumps across from the middle to the side branches of the nullcline are not very sharp. Had we been able to compute the evolution (as μ grows) of the limit cycle for some real small value of $(1-a)$, the departures of the limit cycles from curves made up from parts of the nullcline and horizontal segments would have been indistinguishable. *This is one of those examples where theory and analytical reasoning are much better that a numerical calculation.*

d) Excitable system: $0 < a - 1 \ll 1$.
Consider the situation depicted in figure 5.3-B. If the fixed point is perturbed in such a way that the solution is sent to a point on the phase plane with $y < -2/3$, then the solution will undergo a whole swing around the nullcline,[11] before returning to the critical point. If $a-1$ is small, then the perturbation needed to do this will be small. Since the critical point is a global attractor in this case, it follows that the system is excitable.

In problem 4.5.3 the threshold of excitability was given by the presence of a second critical point. Here there is no other critical point and the role of the second critical point is taken by the the point $(x,\, y) = (1,\, -2/3)$ — notice that, in the limit $\mu \to \infty$ this point behaves very much like a critical point of some strange sort, as can be seen in figure 5.3-B.

6 Problem 08.02.05 - Strogatz (Hopf bifurcation using a computer)

6.1 Statement for problem 08.02.05

For the following system

$$\frac{dx}{dt} = y + \mu\, x \qquad \text{and} \qquad \frac{dy}{dt} = -x + \mu\, y - x^2\, y, \tag{6.1}$$

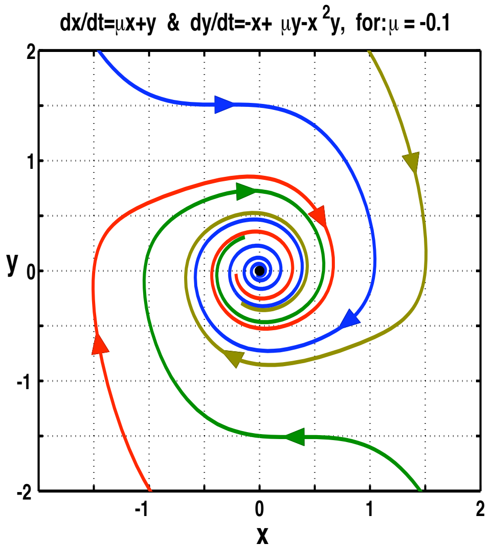
a Hopf bifurcation occurs at the origin when $\mu = 0$. Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of μ, verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius R scales with μ as predicted by theory.

6.2 Answer for problem 08.02.05

See figure 6.1 for the phase portrait with $\mu = -0.1$. It indicates a stabilizing nonlinearity, thus a **supercritical (soft) Hopf bifurcation.**

Figure 6.2 shows a picture of the phase portrait for $\mu = 0.1$ on the left and the limit cycles for various values of $0 < \mu \ll 1$ on the right. A stable limit cycle appears around the critical point for $0 < \mu \ll 1$. This **confirms that a**

[11] Going first to the left branch of the nullcline, then up to the local maximum, etc.



Problem 08.02.05.
System (6.1) for $\mu = -0.1 < 0$.
All the orbits spiral towards the origin, even those $O(1)$ away. Since μ is fairly small, this is a good hint that the nonlinearity is stabilizing, which should lead to a **supercritical (soft) Hopf bifurcation.**

Figure 6.1: (Problem 08.02.05). Phase portrait for the system in (6.1) when $\mu = -0.1 < 0$

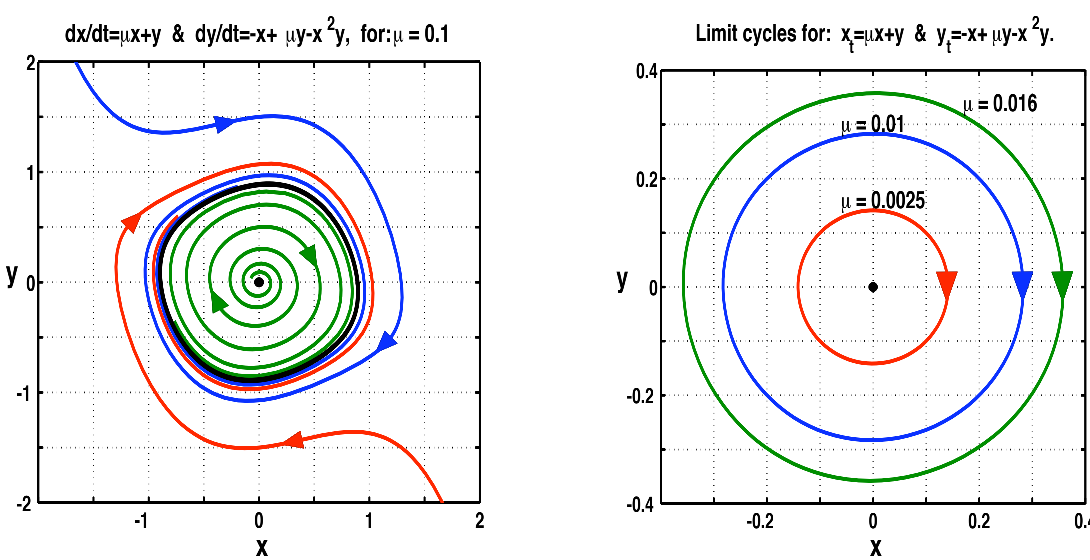


Figure 6.2: (Problem 08.02.05). Left: phase portrait for the system in (6.1) when $\mu = 0.1 > 0$. The picture on the right shows the limit cycles for various values of $0 < \mu \ll 1$.

supercritical (soft) Hopf bifurcation occurs.

For $0 < \mu \ll 1$ it is easy to see (in figure 6.2) that the limit cycles are nearly circular. The table on the right shows a listing of various parameters for these cycles. It should be clear that the theoretical predictions (e.g.: $\dfrac{R}{\sqrt{\mu}} \sim$ constant, and period $\sim$ linear period) are satisfied.

Limit Cycle Parameters, $0 < \mu \ll 1$.

μ	$R =$radius.	$\dfrac{R}{\sqrt{\mu}}$	$\dfrac{\text{Period}}{\pi}$.
0.0160	0.35775	2.8283	2.0014
0.0100	0.28285	2.8285	2.0005
0.0025	0.14142	2.8284	2.0000

Furthermore, the limit cycle for $\mu = 0.1$ is not very circular, but if we interpret its radius as the value of x when $y = 0$, we obtain $R \approx 0.89235$, which yields $\dfrac{R}{\sqrt{\mu}} = 2.8219$ (quite close to the values in the table). In this case the period is

$P \approx 2.0579\pi$.

7 Problem 08.02.07 - Strogatz (Hopf and homoclinic bifurcations using a computer)

7.1 Statement for problem 08.02.07

For the following system

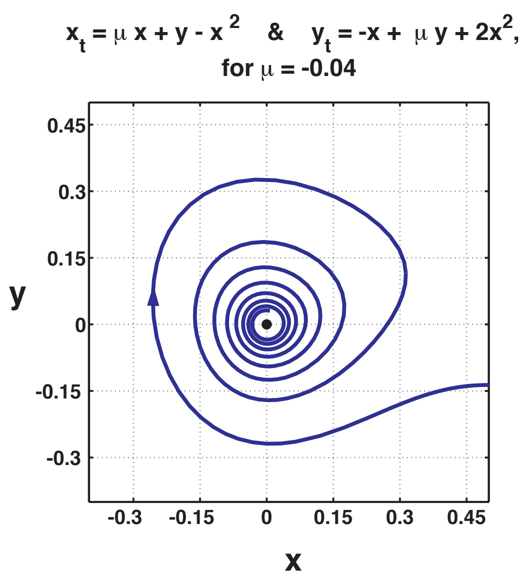
$$\frac{dx}{dt} = \mu\, x + y - x^2 \qquad \text{and} \qquad \frac{dy}{dt} = -x + \mu\, y + 2x^2, \tag{7.1}$$

a **Hopf bifurcation occurs** at the origin when $\mu = 0$. Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of μ, verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius R scales with μ as predicted by theory.

In addition to a Hopf bifurcation, this system also exhibits an **homoclinic bifurcation** of the limit cycle. **FIND IT.**

7.2 Answer for problem 08.02.07

See figure 7.1 for the phase portrait with $\mu = -0.04$. It indicates a stabilizing nonlinearity, thus a **supercritical (soft) Hopf bifurcation.**



Problem 08.02.07.
System (7.1) for $\mu = -0.04 < 0$, when the critical point is a stable spiral.
All the orbits spiral towards the origin, even those $O(1)$ away. Since μ is fairly small, this is a good hint that the nonlinearity is stabilizing, which should lead to a **super-critical (soft) Hopf bifurcation.** See remark **7.4**.

Figure 7.1: (Problem 08.02.07). Phase portrait for the system in (7.1) when $\mu = -0.04 < 0$.

Figure 7.2 shows a picture of the phase portrait for $\mu = 0.04$ on the left and the (stable) limit cycles for various small positive values of $0 < \mu \ll 1$ on the right. A stable limit cycle appears around the critical point for $0 < \mu \ll 1$. Thus **a super-critical (soft) Hopf bifurcation** occurs.

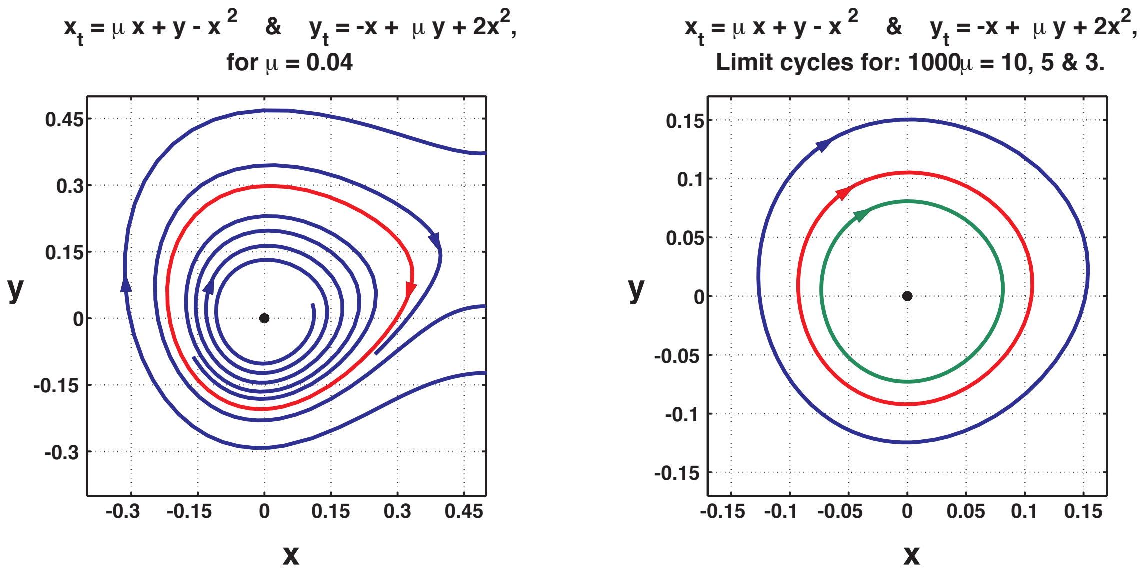


Figure 7.2: (Problem 08.02.07). Left: phase portrait for the system in (7.1) for $\mu = 0.04 < 0$. Right: the limit cycles for three values of $0 < \mu \ll 1$ — larger values of μ correspond to larger limit cycles.

For $0 < \mu \ll 1$ it is easy to see (in figure 7.2) that the limit cycles are nearly circular. The table on the right shows a listing of various parameters for these cycles. It should be clear that the theoretical predictions (e.g.: $\dfrac{R}{\sqrt{\mu}} \sim$ constant, and period $\sim$ linear period) are satisfied.

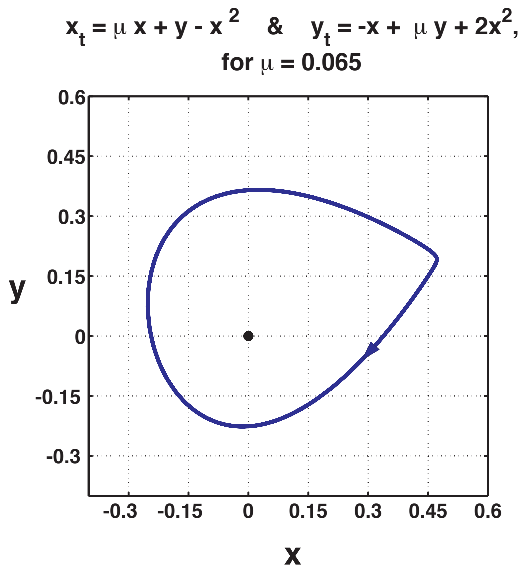
Limit Cycle Parameters, $0 < \mu \ll 1$.

μ	R =radius.	$\dfrac{R}{\sqrt{\mu}}$	$\dfrac{\text{Period}}{2\pi}$.
0.010	0.1259	1.2590	1.0396
0.005	0.0926	1.3096	1.0318
0.003	0.0731	1.3346	1.0113

Furthermore, for this system an homoclinic bifurcation of the limit cycle also happens, as follows: As μ grows from $\mu = 0$, the limit cycle grows in size, till it eventually reaches the critical point at $x = \frac{1+\mu^2}{2+\mu}$ and $y = x^2 - \mu x$ (which is a saddle). At this point an homoclinic bifurcation of the limit cycle happens — see figure 7.3 — with the period of the limit cycle diverging to ∞, and the limit cycle becoming an homoclinic connection for the saddle.

Remark 7.4 *The distinction between a sub and super critical Hopf bifurcation arises* **solely** *from the* **stability** *of the involved limit cycle. If the limit cycle is stable, then the bifurcation is super-critical (or soft). If the limit cycle is unstable, then the bifurcation is sub-critical (or hard). In turn,* this is determined by the role of the nonlinearity at the bifurcation point *(where the linear terms are neutrally stable). If the nonlinear terms are stabilizing, then the bifurcation is super-critical; if the nonlinear terms are destabilizing, then the bifurcation is sub-critical; if the nonlinear terms are neutrally stable, then the situation is singular and the Hopf bifurcation scenario does not apply.*

In the particular case of the system in (7.1), the fact that the limit cycle appears for $\mu > 0$ is **completely irrelevant** to the classification of the type of Hopf bifurcation that occurs. What matters is that the limit cycle that is born is stable.



Problem 08.02.07.
Limit cycle for the system in (7.1) for $\mu = 0.065$, when the limit cycle has grown and it is about to become an homoclinic orbit for the saddle at $\boldsymbol{x = \frac{1+\mu^2}{2+\mu}}$, $\boldsymbol{y = x^2 - \mu x}$. The limit cycle period is about 52 for this value of μ. Note that **an homoclinic bifurcation of this limit cycle is about to happen.**

Figure 7.3: (Problem 08.02.07). Limit cycle for the system in (7.1) when $\mu = 0.65$.

8 Problem 08.04.03 - Strogatz (Homoclinic bifurcation via computer)

8.1 Statement for problem 08.04.03

Using numerical integration, find the value of μ at which the system

$$\frac{dx}{dt} = \mu\, x + y - x^2 \qquad \text{and} \qquad \frac{dy}{dt} = -x + \mu\, y + 2\, x^2, \tag{8.1}$$

undergoes a homoclinic bifurcation. Sketch the phase portrait just above and below the bifurcation. In fact:

1. Find and classify all the critical points for all values of $\boldsymbol{\mu}$.
2. For $\mu = 0$ the origin is a center for the linearized equations. What happens for the nonlinear equations? Are the nonlinear terms stabilizing or destabilizing? What sort of critical point is the origin for the full equations: stable spiral, unstable spiral, or center? **You should be able to do this analytically — See hint 8.1.**
3. What happens at $\boldsymbol{\mu}$ crosses $\mathbf{0}$? (Justify your answer). *The result in item 2 should help here!*
4. Increase μ from $\mu = 0$, and find the homoclinic bifurcation (this is where you'll need a computer).
5. **Optional:** Compute the period of the limit cycle as the homoclinic bifurcation is approached, and verify the theoretical prediction: **period** $\boldsymbol{\sim -\log|\mu - \mu_c|}$.

Remark 8.5 *This problem is very similar (same system of equations) to Strogatz problem 8.2.7.* However: 8.2.7 is purely computational, while here you are being asked to do the analysis behind the problem.

Hint 8.1 *To do the analysis in item* **2**, *you have two alternatives:*

A. *Do a "two-times expansion" for orbits near the critical point. Namely: write the equations in terms of $x = \epsilon\, X$ and $y = \epsilon\, Y$ (where $0 < \epsilon \ll 1$). Then expand.*
B. *Find a "local Liapunov function", $E = (x^2 + y^2)+$ higher order terms, such that $\boldsymbol{\frac{dE}{dt} < 0}$ near the origin. In fact $\frac{dE}{dt} \leq 0$ is O.K., as long as $\frac{dE}{dt} = 0$ only for curves the orbits cross — e.g. the axis.*

The first alternative is a straightforward application of the methods in the "Weakly Nonlinear Things" notes. The second actually provides a rigorous proof of the result. However, **it turns out that getting $\boldsymbol{E}$ is not completely trivial!** *The naive approach to searching for E is*

0. *Define $E_0 = x^2 + y^2$ and compute its time derivative. This yields*
$$\frac{dE_0}{dt} = \text{(3rd-order terms)} + \text{(4th-order terms)}.$$
Of course, this is not good enough: the 3rd-order terms can have any sign. Hence:
1. *Add 3rd-order term "corrections" to E_0, to eliminate the 3rd-order terms in $\dot{E}_0$. That is, define $E_1 = E_0+$ 3rd-order terms, so that*
$$\frac{dE_1}{dt} = \text{(4th-order terms)} + \text{(5th-order terms)}.$$
There is only one way to do this. Unfortunately, some of the 4th-order terms are positive. Hence:
2. *Add 4th-order terms "corrections" to E_1, to eliminate the bad 4th-order terms in $\dot{E}_1$. That is, define $E_2 = E_1+$ 4th-order terms, so that*
$$\frac{dE_2}{dt} = \text{(negative 4th-order terms)} + \text{(5th-order terms)} + \text{(6th-order terms)}.$$
Again: there is only one way to do this. Unfortunately, this still does not work. Some of the higher order terms here are always smaller than the negative 4-th order terms, but some are not. For example, if $-x^2\,y^2$ is a negative 4-th order term, then: (i) $-x^2\,y^2 + x^3\,y^2$ is always negative for $x^2 + y^2 \ll 1$, so $x^3\,y^2$ is not a problem, but (ii) $-x^2\,y^2 + x^4\,y$ can switch sign (if $0 < y < x^2 \ll 1$), so $x^4\,y$ is a "bad" term. Hence:
3. *Add 5th-order terms "corrections" to E_2, to eliminate the bad 5th order terms ... Unfortunately, you then end up with "bad" 6th order terms!*

This never ends! Fortunately: *if you do the process above correctly, you will notice that: while the terms in E_n involve ever higher powers of y, there is only a very small set of powers of x that appear. Hence, look for a Liapunov function of the form $\boldsymbol{E = g(y) + x^2\,f(y) + \dots}$, where g, f, etc., are to be determined. This will work: there is a finite (and small) numbers of terms involved. After you have obtained E in this fashion, you will see that it can be expanded as in item B above.*

8.2 Answer for problem 08.04.03

Before we start, let us find the critical points for the system in (8.1). Multiplying the first equation by μ, and subtracting from the result to the second equation, yields $0 = (1+\mu^2)\,x - (2+\mu)\,x^2$ at the critical points. From this, and $y = -\mu\,x + x^2$ (also valid at the critical points), we obtain:

- $\boldsymbol{P_0 = (x,\, y) = (0,\, 0)}$ is a **stable spiral** for $\mu < 0$, and an **unstable spiral** for $\mu > 0$. It can be shown (see § 8.2.1) that **a supercritical (soft) bifurcation occurs for $\boldsymbol{\mu = 0}$, and a stable limit cycle appears enclosing the origin for $\boldsymbol{\mu > 0}$.**
- $\boldsymbol{P_s = (x,\, y) = \frac{1+\mu^2}{2+\mu}\left(1,\ \frac{1-2\,\mu}{2+\mu}\right)}$ is a critical point for $\mu \neq -2$. Always a **saddle**.

The **homoclinic bifurcation** occurs as the limit cycle created by the Hopf bifurcation at $\mu = 0$ grows in size with μ, till at a **critical $\boldsymbol{\mu = \mu_c \approx}$ 0.0661** it collides with the saddle at P_s, and it is destroyed. This process is illustrated by figures 8.1 and 8.2.

On the left in figure 8.1, for $\mu = 0.055$, we see how the limit cycle has grown from its original position close to the origin, and it is now large enough to get very close to the saddle at P_s. Notice how the stable manifold for the saddle approaches the limit cycle, while the unstable one "hugs" it closely (going backwards in time) and then leaves towards infinity.

On the right in figure 8.1, for $\mu = 0.066$, only the limit cycle is shown. This μ is very close to the critical value. The limit cycle is very close to the saddle, and is very hard to distinguish it from an homoclinic connection. Precisely at $\mu = \mu_c$ the limit cycle becomes an homoclinic connection, with the period going to infinity. As $\mu \to \mu_c$, the limit cycle period behaves like $|\log(\mu_c - \mu)|$ — see the table below.

Finally, figure 8.2 shows the phase portrait for $\mu = 0.077$, above critical. The limit cycle has disappeared, the left un-stable manifold from the saddle approaches the spiral point as $t \to -\infty$, while the stable one approaches infinity as $t \to \infty$.

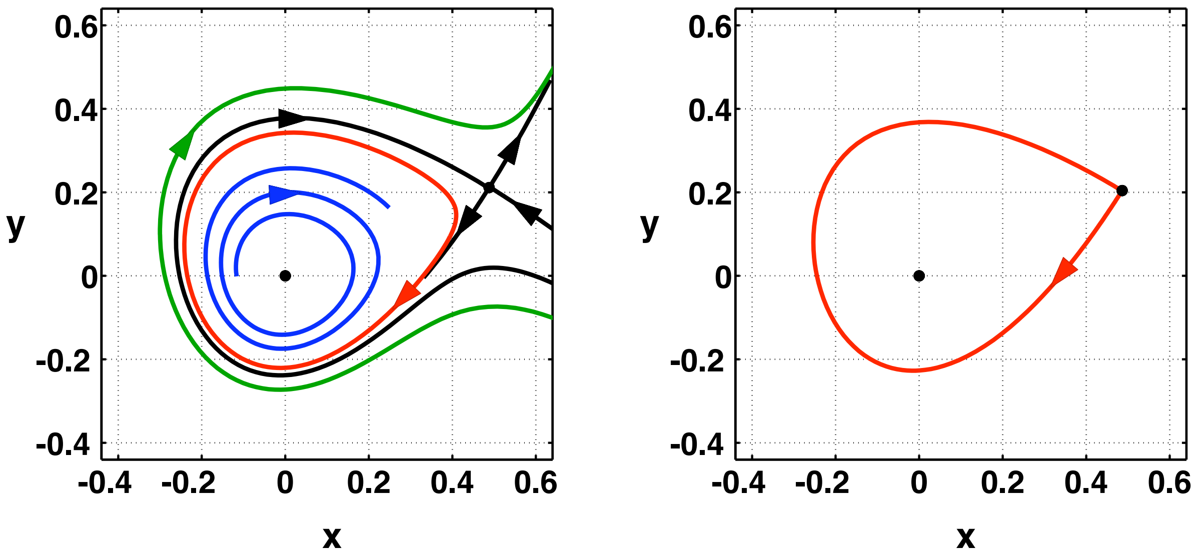
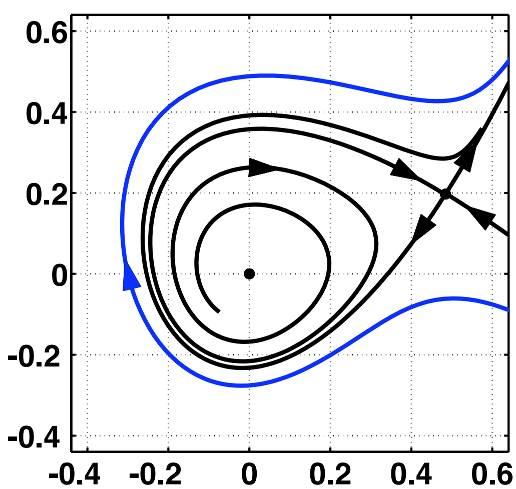


Figure 8.1: (Problem 8.4.3). Phase portraits for the system (8.1), $\dot{x} = \mu\, x + y - x^2$ & $\dot{y} = -x + \mu\, y + 2\, x^2$. Left: $\mu = 0.055$, slightly below $\mu_c \approx 0.0661$. Right: limit cycle for $\mu = 0.066$, below but very close to μ_c.



Phase plane portrait for the system in (8.1), that is
$$\dot{x} = \mu\, x + y - x^2 \quad \text{and} \quad \dot{y} = -x + \mu\, y + 2\, x^2,$$
with $\mu = 0.077$. This value is slightly above the value $\mu_c \approx 0.061$, at which an homoclinic bifurcation of a limit cycle occurs.
Horizontal axis = x. Vertical axis = y.

Figure 8.2: (Problem 8.4.3). Phase portrait for (8.1) with $\mu = 0.077$, slightly above $\mu_c \approx 0.0661$.

In general, homoclinic bifurcations are hard to find by means other than numerical integration of the equations — this example is no exception to this rule. Perhaps there is an argument one can make to find (and approximate) the critical value at which the bifurcation occurs, but I was unable to find one. I was not even able to produce an argument indicating that a bifurcation should occur, other than the following (very gross) one: Once the Hopf bifurcation occurs, the size of the limit cycle grows like $\sqrt{\mu}$. Since the saddle starts fairly close to the origin, even rather small values of μ give values of $\sqrt{\mu}$ that make the limit cycle large enough to "reach" the saddle. Hence, it is not unreasonable to expect a bifurcation to occur for some small μ. This is what happens, with $\sqrt{\mu_c} \approx 0.26$ and $P_s(\mu_c)$ at a distance ≈ 0.53 from P_0.

For $0 < |\mu - \mu_c| \ll 1$ the period of a limit cycle about to disappear (due to an homoclinic bifurcation) behaves like $-\log(|\mu - \mu_c|)$. The table on the right illustrates this, with $\alpha = \frac{\text{Period}}{-\log(\mu_c - \mu)}$ = constant at leading order.

Note that the calculation of μ_c is not very reliable, so getting very close to it is not quite possible. Nevertheless, the table shows reasonable agreement with the theoretical expectation.

μ	Period	α
0.055	8.969	1.993
0.057	9.265	1.971
0.059	9.636	1.948
0.061	10.130	1.919
0.063	10.877	1.883
0.065	12.458	1.829

8.2.1 Analysis for the Hopf bifurcation

The origin switches from a stable to an unstable spiral point as μ crosses $\mu = 0$. Hence, in order to show that a supercritical Hopf bifurcation occurs for $\mu = 0$, all we need to do is to show that the origin is (nonlinear) stable spiral for $\mu = 0$. Namely, for the system

$$\frac{dx}{dt} = y - x^2 \qquad \text{and} \qquad \frac{dy}{dt} = -x + 2\,x^2. \tag{8.2}$$

Following the hint (I will not display here the calculations described in the hint that motivate this form), we search for a Liapunov function of the form

$$\boldsymbol{E = g(y) + x^2\, f(y) + x^3\, h(y).} \tag{8.3}$$

It is then easy to check that

$$\frac{d}{dt}\,(g) = -x\,g' + 2\,x^2\,g', \tag{8.4}$$

$$\frac{d}{dt}\left(x^2\,f\right) = 2\,x\,y\,f - 2\,x^3\,f - x^3\,f' + 2\,x^4\,f', \tag{8.5}$$

and

$$\frac{d}{dt}\left(x^3\,h(y)\right) = 3\,x^2\,y\,h - 3\,x^4\,h - x^4\,(1 - 2\,x)\,h'. \tag{8.6}$$

The terms linear in x cancel if $g' = 2\,y\,f$, and the cubic ones cancel if $f' = -2\,f$. Thus

$$\boldsymbol{f = e^{-2\,y} > 0} \quad \text{and} \quad \boldsymbol{g = \frac{1}{2}\left(1 - e^{-2\,y}\right) - y\,e^{-2\,y}} \tag{8.7}$$

yield

$$\frac{dE}{dt} = 4\,x^2\,y\,f - 4\,x^4\,f + 3\,x^2\,y\,h - 3\,x^4\,h - x^4\,(1 - 2\,x)\,h'. \tag{8.8}$$

Now take $\boldsymbol{h = -\frac{4}{3}\,f}$, to obtain

$$\boldsymbol{\frac{dE}{dt} = -\frac{8}{3}\,x^4\,(1 - 2\,x)\,f.} \tag{8.9}$$

This is precisely what we were looking for. For x small, $\boldsymbol{\dot{E} < 0}$, except along the coordinate line $x = 0$ where $\boldsymbol{\dot{E} = 0}$. Since non-trivial orbits cross $x = 0$ (i.e.: $\dot{x} \neq 0$ for $x = 0$ and $y \neq 0$) E is a decreasing function along them. Further:

$$E = x^2 + y^2 + O((x^2 + y^2)^{3/2}), \tag{8.10}$$

so that the origin is a local minimum for E. In addition, it is easy to see that $\dot{\theta} = -1 + O((x^2 + y^2)^{1/2})$, where θ is the polar angle. **Hence the origin is a stable spiral point.**

THE END.