Answers to P-Set # 04, 18.385j/2.036j
MIT (Fall 2020)

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October 22, 2020

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1.1 Statement for problem 14.09.22

Recall the *definitions for the various types of stability* that concern critical points:
Let $\mathbf{x}^*$ be a fixed point of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then:

1. $\mathbf{x}^*$ is **attracting** if there is a $\delta > 0$ such that $\lim_{t\to\infty} = \mathbf{x}^*$ whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. That is: any trajectory that starts within δ of $\mathbf{x}^*$ *eventually* converges to $\mathbf{x}^*$. Note that trajectories that start nearby $\mathbf{x}^*$ *need not stay close in the short run*, but *must approach* $\mathbf{x}^*$ *in the long run.*
2. $\mathbf{x}^*$ is **Liapunov stable** if for each $\epsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for $t > 0$, whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. Thus, trajectories that start within δ of $\mathbf{x}^*$ stay within ϵ of $\mathbf{x}^*$ for all $t > 0$.
   In contrast with attracting, Liapunov stability requires nearby trajectories to remain close *for all* $t > 0$.
3. $\mathbf{x}^*$ **is asymptotically stable** if it is *both* attracting and Liapunov stable.
4. $\mathbf{x}^*$ is **repeller** if there exist $\epsilon > 0$ and $\delta > 0$ such that: if $0 < \|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then (after some critical time) it will be $\|\mathbf{x}(t) - \mathbf{x}^*\| > \epsilon$ (i.e., for $t > t_c$). Repellers are a special kind of *unstable* critical points.

For each of the following systems, decide whether the origin is attracting but not Liapunov stable, Liapunov stable, asymptotically stable, repeller, or unstable but not a repeller.

a) $\dot{x} = 2\,y$ and $\dot{y} = -3\,x$.
b) $\dot{x} = y\,\cos(x^2 + y^2)$ and $\dot{y} = -x\,\cos(x^2 + y^2)$.
c) $\dot{x} = -x$ and $\dot{y} = -|y|\,y$.
d) $\dot{x} = 2\,x\,y$ and $\dot{y} = y^2 - x^2$. *Hint: what happens along $x = 0$?*
e) $\dot{x} = x - 2\,y\,x^2 - 4\,y^3$ and $\dot{y} = y + x^3 + 2\,x\,y^2$.
f) $\dot{x} = y$ and $\dot{y} = x$.
g) Finally, consider the critical point $(x,\, y) = (1,\, 0)$, for the system

$$\dot{x} = (1 - r^2)\,x - (1 - \tfrac{x}{r})\,y \quad \text{and} \quad \dot{y} = (1 - r^2)\,y + (1 - \tfrac{x}{r})\,x, \tag{1.1}$$

defined in the “punctured” plane $r = \sqrt{x^2 + y^2} > 0$. *Hint: write the equations in polar coordinates.*

Additional hints. In some cases you can get the answer by finding a function $\mathcal{J} = \mathcal{J}(x,\, y)$ with a local minimum at the origin such that $\frac{d\mathcal{J}}{dt} > 0$ along trajectories — or maybe one such $\frac{d\mathcal{J}}{dt} < 0$, or maybe one such $\frac{d\mathcal{J}}{dt} = 0$. In other cases look for special trajectories that either leave, or approach, the origin.

1.2 Answer for problem 14.09.22

These are the answers, for each system:

a) It is easy to check that $\mathcal{J} = 3\,x^2 + 2\,y^2$ is constant along the trajectories.
The origin is a center, thus **Liapunov stable** — it is, in fact, a **linear, stable, node.**

b) It is easy to check that $\mathcal{J} = x^2 + y^2$ is constant along the trajectories.
The origin is a center, thus **Liapunov stable.**

c) In terms of the initial values, the solution for $t > 0$ is $x = x_0\,e^{-t}$ and $y = y_0/(1 + |y_0|\,t)$.
The origin is **asymptotically stable** — it is, in fact, a **nonlinear, stable, node.**

d) The first equation shows that, if x vanishes anywhere, then it vanishes everywhere — i.e., $x = 0$ is an invariant curve. The other equation then reduces to $\dot{y} = y^2$. Thus $x = 0$ and $y > 0$ is an orbit leaving the origin, and $x = 0$ and $y < 0$ is an orbit approaching the origin.
The origin is **unstable, but not a repeller.**
Note: for this system it can be shown that all the trajectories that have $x \neq 0$ somewhere (thus everywhere, why?) approach the origin as $t \to \infty$.

e) It is easy to check that $\mathcal{J} = x^2 + 2\,y^2$ satisfies $\frac{d\mathcal{J}}{dt} = 2\,\mathcal{J}$ along the trajectories.
The origin is a **repeller.**
Note: in fact, the origin is a nonlinear, unstable, spiral. This follows from the fact that the equations yield, for the polar angle θ, $\dot{\theta} = \mathcal{J}/r^2$ — where $r^2 = x^2 + y^2$.

f) It is easy to check that $\mathcal{J} = x^2 - y^2$ is constant along the trajectories.
The origin is a saddle (in fact, a linear saddle), thus **unstable, but not a repeller.**

g) In polar coordinates equation (1.1) takes the form

$$\dot{r} = (1 - r^2)\,r \quad \text{and} \quad \dot{\theta} = 1 - \cos\theta. \tag{1.2}$$

From this it should be clear that, for any initial data $0 < r_0$ and $0 < \theta < 2\,\pi$, the solution approaches the critical point as $t \to \infty$. Furthermore, the real positive axis (i.e., $\theta = 0$) is an invariant curve, and along it the system reduces to $\dot{x} = (1 - x^2)\,x$ — thus, again, the trajectories approach the critical point as $t \to \infty$. It follows that **the critical point is attracting.**

On the other hand, consider the trajectory $r \equiv 1$ and $0 < \theta < 2\,\pi$. This trajectory *starts* at the critical point at $t = -\infty$, goes around the unit circle counterclockwise, and arrives back at the critical point at $t = \infty$. In fact, any solution starting near the critical point with $y > 0$, initially moves away from the critical point (as far as a distance ≈ 2), before returning to the critical point as $t \to \infty$. Thus **the critical point is not Liapunov stable.**

2 Problem 06.01.10 - Strogatz (Computer generated phase portrait)

2.1 Statement for problem 06.01.10

First, plot a computer generated phase plane portrait for the ”two-eyed monster”

$$\frac{dx}{dt} = y + y^2 \qquad \text{and} \qquad \frac{dy}{dt} = -\frac{1}{2}\,x + \frac{1}{5}\,y - x\,y + \frac{6}{5}\,y^2. \tag{2.1}$$

In particular, make a plot that covers the region $-5 \leq x \leq 3$ and $-3 \leq y \leq 2$.
Next find the critical points and classify them. Does what you observe in the plot match what the theory predicts? Explain any discrepancies. *Hint: Explore carefully what happens close to the critical points.*

2.2 Answer for problem 06.01.10

Figure 2.1 (left) shows a computer generated phase portrait for the solutions of (2.1). This is a "large" scale picture and (to avoid crowding) we have not indicated the orbit directions with arrows.

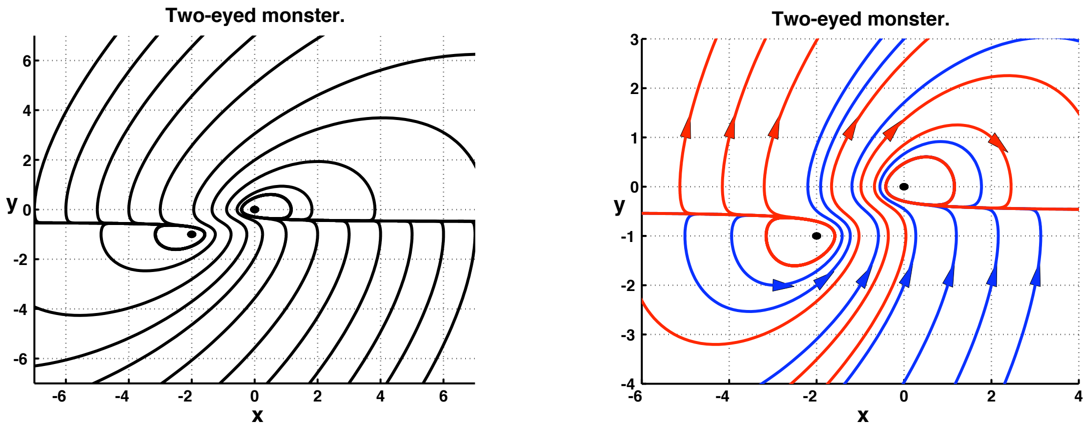

Figure 2.1: (Problem 06.01.10). "Two-eyed monster". Critical points marked by the dots at the eye's pupils.

Figure 2.1 (right) shows a smaller region of the phase plane, with the flow directions indicated by arrows. Note that something "funny" is going on near the critical points, as we explain next

The critical points of equation (2.1) are $\boldsymbol{P_l = (-2,\ -1)}$ and $\boldsymbol{P_r = (0,\ 0)}$. It is clear, from a linearized analysis, that

A. $\boldsymbol{P_l}$ **is a stable spiral, with eigenvalues** $\boldsymbol{\lambda = -0.1 \pm i\sqrt{0.49}}$**.**

B. $\boldsymbol{P_r}$ **is an un-stable spiral, with eigenvalues** $\boldsymbol{\lambda = +0.1 \pm i\sqrt{0.49}}$**.**

On the other hand, the picture in figure 2.1 (right) gives the impression that the orbits are leaving P_l and approaching P_r, which certainly contradicts **A** and **B** above. *How do we explain this?*

The answer to the mystery in the prior paragraph is provided by figure 2.2, which shows a detail near P_r. The

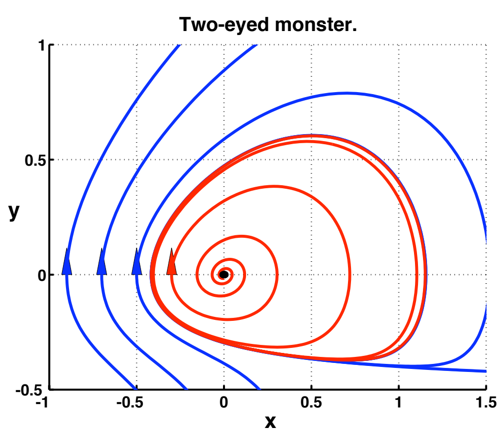

Figure 2.2: (Problem 06.01.10). Detail near the critical point P_r (un-stable spiral), marked by the dot at.

picture shows that, indeed, the critical point is an un-stable spiral, but it is enclosed by a stable limit cycle. Thus, *all the orbits that seem to be approaching* P_r *in figure* 2.1 *(right), in fact are approaching this limit cycle.* Similarly, *there is an un-stable limit cycle* near P_l, so that the orbits that seem to be leaving P_l, are in fact leaving the unstable limit cycle near P_l (the phase plane portrait near P_l looks quite similar to the picture shown in figure 2.2, except that the orbits move away from the limit cycle, rather than towards it).

In summary, this is the global behavior for all the orbits:

1. Orbits starting within the limit cycle enclosing P_l, approach P_l as $t \to \infty$ and the limit cycle as $t \to -\infty$.
2. Orbits starting within the limit cycle enclosing P_r, approach P_r as $t \to -\infty$ and the limit cycle as $t \to \infty$.
3. All other orbits approach the limit cycle enclosing P_r (resp. P_l) as $t \to \infty$ (resp. $t \to -\infty$).

Remark 2.1 *Explain why all the orbits, for $|x| \gg 1$, seem to either approach a curve with $y \approx -0.5$ (for $x > 0$) or leave it (for $x < 0$).* **(Extra task #1).**

3 Problem 06.02.02 - Strogatz (A trapped solution)

3.1 Statement for problem 06.02.02

Consider the system

$$\left.\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x + (1 - x^2 - y^2)\, y. \end{aligned}\right\} \tag{3.1}$$

a. Let $\boldsymbol{D}$ be the open disk $x^2 + y^2 < 4$. Verify that the system satisfies the hypothesis of the existence and uniqueness theorem [1] throughout the domain $\boldsymbol{D}$.

b. By substitution, show that $x(t) = \sin(t)$ and $y(t) = \cos(t)$ is an exact solution of the system.

c. Now consider a different solution, in this case starting from the initial conditions

$$x(0) = \frac{1}{2} \quad \text{and} \quad y(0) = 0.$$

Without doing any calculations, explain why this solution *must* satisfy $x(t)^2 + y(t)^2 < 1$ for all $t < \infty$.

3.2 Answer for problem 06.02.02

a. The hypothesis we must verify are those stated in the theorem at the start of section 6.2 of the book. Now: The set $\boldsymbol{D}$ is open and connected, and the right hand side of the system in (3.1) is in fact analytic (a polynomial in the variables), which is far more than needed.

b. This part is rather trivial, since $x^2 + y^2 \equiv 1$ for the given solution.

c. Since $x(0)^2 + y(0)^2 = 0.25 < 1$, uniqueness (see part **a**) guarantees that $x(t)^2 + y(t)^2 < 1$ for all times. Otherwise this solution would have to cross the solution given in part **b**, and there would be two different solutions going through the same point in phase space.

[1] See § 6.2 of Strogatz book — p. 149 (1st edition), p. 150 (2nd edition). A stronger version was stated during Lectures 1-2.

4 Problem 06.03.11 - Strogatz (A nonlinearity changes a star into a spiral)

4.1 Statement for problem 06.03.11

Here is an example showing that borderline fixed points are sensitive to nonlinear terms. Consider the system in polar coordinates given by

$$\frac{dr}{dt} = -r \qquad \text{and} \qquad \frac{d\theta}{dt} = \frac{1}{\ln(r)}, \quad \text{where } 0 \leq r < 1. \tag{4.1}$$

a) Write the system in x, y coordinates, in the form $\dfrac{dx}{dt} = f(x,\, y)$ and $\dfrac{dy}{dt} = g(x,\, y)$.

b) Show that the linearized system about the origin is $\dfrac{dx}{dt} = -x$ and $\dfrac{dy}{dt} = -y$.
Thus the origin is a stable star for the linearization. See remark **4.2**.

c) Find $r = r(t)$ and $\theta(t)$ explicitly, given initial conditions $(r_0,\, \theta_0)$ — with $0 < r_0 < 1$.

d) Show that the solutions in item **c** satisfy $r(t) \to 0$ and $\theta(t) \to -\infty$ as $t \to \infty$. Therefore the origin is a stable spiral for the nonlinear system.

Remark 4.2 *This system is not smooth at the origin, since both f and g fail to have second derivatives there. Nevertheless, both have continuous partial derivatives for $r < 1$. Hence, (i) a linearized system at the origin is well defined, (ii) the theorem guaranteeing existence and uniqueness for the solutions applies.*

4.2 Answer for problem 06.03.11

Part (a)
From $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have: $\dot{x} = \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta}$ and $\dot{y} = \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta}$. Thus:

$$\dot{x} = -x - \frac{2y}{\ln(x^2+y^2)} \qquad \text{and} \qquad \dot{y} = -y + \frac{2x}{\ln(x^2+y^2)}. \tag{4.2}$$

Part (b)

Linearizing (4.2) about the origin gives the system on the right (note that the flow vector is not smooth at the origin, nevertheless: it still has a linearization, with the error term $O(r/\ln(r))$, instead of the usual $O(r)$ for smooth systems).

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This system has a stable star node (see page 135 of Strogatz book, figure 5.2.5).

Part (c)
Here we find the solution of (4.1), with initial conditions $0 < r(0) = r_0 < 1$ and $\theta(0) = \theta_0$, explicitly. To do this, we fist solve the equation for $r = r(t)$ and substitute the answer into the equation for $\theta = \theta(t)$, which we then solve. This yields:

$$r(t) = r_0 e^{-t} \qquad \text{and} \qquad \theta(t) = -\ln(t - \ln r_0) + \ln(-\ln(r_0)) + \theta_0, \tag{4.3}$$

which is defined for $t > \ln(r_0)$ only. Note that $t = \ln(r_0) < 0$ is the time when the solution to (4.1) — evolving backwards in time — reaches $r = 1$, where the system is singular.

Part (d)
From (4.3) we see that $r(t) \to 0$ and $\theta(t) \to -\infty$ as $t \to \infty$. Thus **the origin is a stable spiral point for this nonlinear system.**

5 Problem 06.03.13 - Strogatz (A linear center that is a nonlinear spiral)

5.1 Statement for problem 06.03.13

(Another linear center that is actually a nonlinear spiral). Consider the system

$$\frac{dx}{dt} = -y - x^3 \qquad \text{and} \qquad \frac{dy}{dt} = x. \tag{5.1}$$

Show that the origin is a spiral, although the linearization predicts a center.

5.2 Answer for problem 06.03.13

The linearized equations near the (single) critical point at the origin are

$$\frac{dx}{dt} = -y \qquad \text{and} \qquad \frac{dy}{dt} = x. \tag{5.2}$$

These, obviously, have a center at the origin (they are, essentially, the harmonic oscillator equations). On the other hand, for the full system, we have:

$$\frac{dE}{dt} = -x^4 \leq 0, \tag{5.3}$$

where $\boldsymbol{E = \frac{1}{2}(x^2 + y^2)}$ is the "energy". This is not quite a Liapunov function (by the book's definition) because $\frac{dE}{dt}$ vanishes for $x = 0$. However, for any non-trivial orbit, $\frac{dx}{dt} \neq 0$ when $x = 0$, so that we can conclude that E **is a (strictly) monotone decreasing function of t along any orbit.** It follows that

A. All orbits approach the origin as $t \to \infty$.

On the other hand, using polar coordinates ($x = r\cos(\theta)$ and $y = r\sin(\theta)$), it is clear that

$$\frac{d\theta}{dt} = 1 + r^2 \sin(\theta)\cos^3(\theta), \quad \text{so that}$$

B. For $r < 1$, θ is a monotone increasing function of t.

From **(A)** and **(B)**, it immediately follows that the origin is a spiral point.

Remark 5.3 *Eliminating the variable x in (5.1) yields $\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + y = 0$. Thus, (5.1) is the harmonic oscillator with the nonlinear damping coefficient given by $\left(\frac{dy}{dt}\right)^2$.*

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6 Problem 06.05.07 - Strogatz (General relativity and planetary orbits)

6.1 Statement for problem 06.05.07

The relativistic equation for the orbit of a planet around the Sun is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \epsilon\, u^2, \tag{6.1}$$

where $u = 1/r$ and $(r,\, \theta)$ are the polar coordinates for the planet's position in the plane of motion. The parameter α is positive and can be found explicitly from classical Newtonian mechanics.[2] The term $\epsilon\, u^2$ is Einstein's correction, where ϵ is a very small positive parameter.

[2] The parameter α is related to the angular momentum of the orbit.

a) Rewrite the equation as a system in the $(u,\, v)$ plane, where $v = du/d\theta$.

b) Find all the equilibrium points of the system.

c) Show that one of the equilibria is a center in the $(u,\, v)$ phase plane, according to the linearization. Is it a *nonlinear* center?

d) Show that the equilibrium point found in **(c)** corresponds to a circular planetary orbit.

e) **Extra questions, not in the book version.** The equation has solutions where u is a periodic function of θ. **(1)** *Do this solutions correspond to periodic orbits around the Sun?* **(2)** *If not, what do they correspond to?* **(3)** *What happens when $\epsilon = 0$ (Newtonian mechanics)?*

6.2 Answer for problem 06.05.07

a) Introducing $v = du/d\theta$, the equation can be written as the phase plane system:

$$\left.\begin{aligned} \frac{du}{d\theta} &= v, \\ \frac{dv}{d\theta} &= -u + \alpha + \epsilon\, u^2. \end{aligned}\right\} \tag{6.2}$$

b) The equilibrium points for the system in (6.2) are given by $v = 0$ and

$$u = \frac{1}{2\epsilon}\left\{1 \pm \sqrt{1 - 4\,\epsilon\,\alpha}\right\}.$$

Thus, either $\boldsymbol{u = u_1 \sim \frac{1}{\epsilon} - \alpha + \dots}$ or $\boldsymbol{u = u_2 \sim \alpha + \epsilon\,\alpha^2 \dots}$

c) Note that the equation in (6.1) is a conservative system, with energy $\boldsymbol{E = \frac{1}{2}\,v^2 + V(u)}$, where the potential V is given by $\boldsymbol{V = \frac{1}{2}\,u^2 - \alpha\,u - \frac{1}{3}\,\epsilon\,u^3}$. Furthermore, u_2 is a local minimum for this potential, while u_1 is a local maximum. It follows that, for the two critical points in part **(b)** above, we have:
c1. $(u,\, v) = (u_1, 0)$ is a saddle.
c2. $(u,\, v) = (u_2, 0)$ is a center.

d) The equilibrium points have $u = 1/r$ constant, thus they correspond to circular orbits.

Remark 6.4 *Near the critical point $(u,\, v) = (u_2, 0)$, the solutions of the system in (6.2) are all periodic functions of θ. Does this mean then that the planet orbits around the Sun following a closed periodic orbit? The answer to this is no, unless the period of the solution of (6.2) is 2π (and u does not vanish on the orbit). Otherwise, as the planet goes around the Sun one turn, it does not return to its prior position, but some other and the orbit does not quite close.*
For $\epsilon = 0$ (Newtonian Mechanics) the solutions of (6.1), given by

$$u = \alpha + c\,\sin(\theta + \theta_0),$$

are all periodic of period 2π, and the orbits with $\alpha > |c|$ are all closed and periodic (Kleper ellipses). *In the Relativistic case, however, the period is a function of the amplitude of the deviation from the center, and the orbits are then not closed (for small enough deviations, they look like ellipses whose principal axes rotate slowly in space).*

7 Problem 06.08.09 - Strogatz (Counter-rotating limit cycles)

7.1 Statement for problem 06.08.09

A smooth vector field on the phase plane is known to have exactly two closed trajectories, one of which lies inside the other. The inner circle runs counterclockwise, and the outer runs clockwise.
True or False: *there must be at least one fixed point in the region between the cycles.*
If true, prove it. If false, provide a simple counterexample.
Hint: Beware of "gut feeling" instinctive answers. There is a good chance that your intuition is wrong!

7.2 Answer for problem 06.08.09

False. Consider systems of the form

$$\frac{dx}{dt} = a\,x - b\,y \quad \text{and} \quad \frac{dy}{dt} = b\,x + a\,y, \tag{7.1}$$

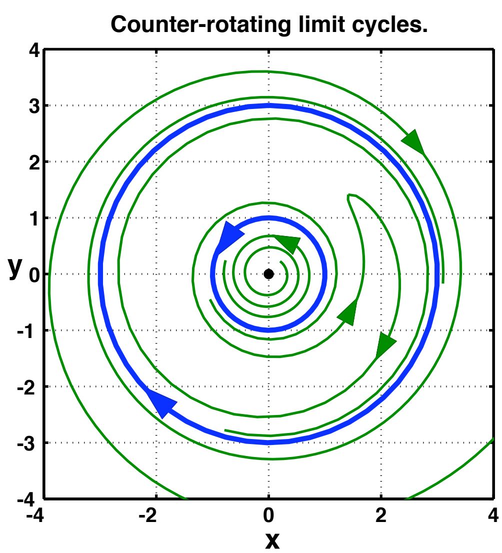
where a and b are smooth functions of the square of the radius, $r^2 = x^2 + y^2$. Namely: $a = a(r^2)$ and $b = b(r^2)$. The system in (7.1) yields a smooth vector field. In polar coordinates this is the same as:

$$\frac{dr}{dt} = a\,r \qquad \text{and} \qquad \frac{d\theta}{dt} = b. \tag{7.2}$$

Now select the functions a and b so that

A. $a = a(r^2)$ has exactly two zeros, $r^2 = a_1 > 0$ and $r^2 = a_2 > a_1$. e.g.: $a = (1 - r^2)(r^2 - 3)$.

B. $b = b(r^2)$ is such that $b(a_1) > 0$ and $b(a_2) < 0$. e.g.: $b = 2 - r^2$.



Here $a = 0.03\,(1 - r^2)(r^2 - 9)/(3 + 2r^2)$
and $b = \arctan(0.3\,(4 - r^2))$.
These values are selected to get plots where the structure is clearly visible (e.g. avoid "crowded" spirals near the critical point). The two limit cycles, the critical point at the origin, and one typical orbit in each region are shown. Other orbits follow by rotation of these, since the system is invariant under rotation. The arrows indicate the flow direction.

Figure 7.1: (Problem 06.08.09). Phase plane portrait for equation (7.1).

Any system of the type above provides a counterexample, since

- Exactly one critical point: the origin — a stable or unstable spiral depending on the sign of $a(0)$. No other point is a fixed point, since nowhere for $r > 0$ do both $\dot{r}$ and $\dot{\theta}$ vanish simultaneously.
- Exactly two closed orbits: the circles $r \equiv \sqrt{a_1}$ (runs counterclockwise) and $r \equiv \sqrt{a_2}$ (runs clockwise).

Figure 7.1 shows the phase portrait for a system of this type.

8 Problem 07.02.17 - Strogatz (Dulac's criterion on an annulus)

8.1 Statement for problem 07.02.17

Assume the hypothesis of Dulac's criterion. However, assume that the region of interest, R, is (instead of simply connected) topologically equivalent to an annulus (i.e.: it has exactly one hole in it). Using Green's theorem, show that there exists *at most* one closed orbit in R — *this result can be useful as a way of proving that a closed orbit is unique.*

8.2 Answer for problem 07.02.17

We begin by considering the possible ways in which a limit cycle can fit inside the region R. It is easy to see that either:

A. The **limit cycle does not enclose the hole in the region,** as in the example of the limit cycle Γ_2 in the left panel of figure 8.1. It is clear that a cycle of this type can always be enclosed in a simply connected sub–region of R. Then Dulac's criteria tells us that **such a cycle is not possible.**

B. The **limit cycle encloses the hole in the region,** as in the example of the limit cycle Γ_1 in the left panel of figure 8.1. Such a cycle can only go once around the hole (it cannot have self-crossings). We show below that, **with the hypothesis above, there can be at most one such limit cycle.**

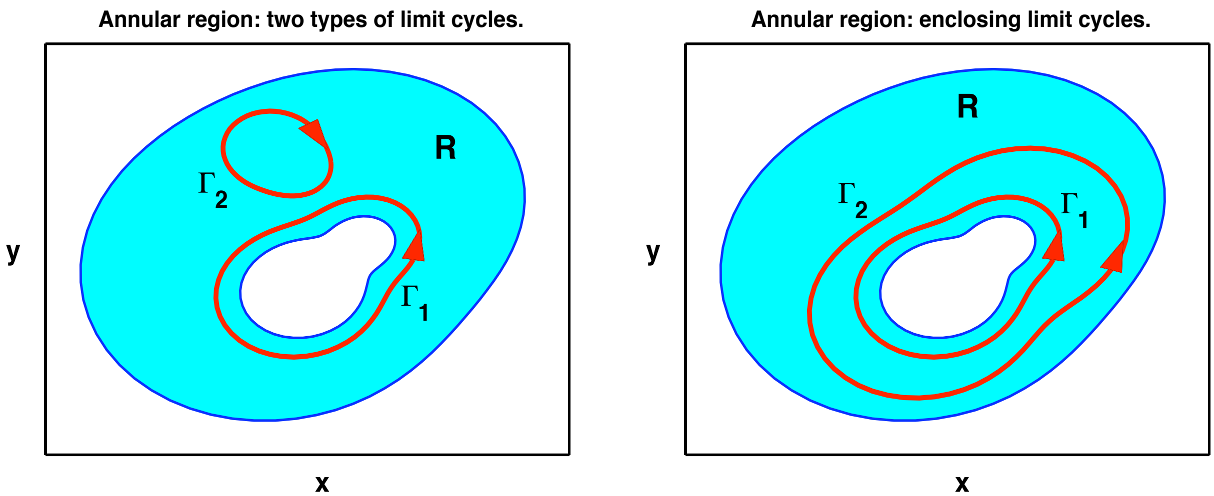


Figure 8.1: (Problem 07.02.17). The left panel illustrates the two possibilities for limit cycles in an annular region: either the cycle encloses the hole, Γ_1, or it does not, Γ_2. The right panel shows two cycles enclosing the hole in the region. They must also enclose each other, since orbits cannot cross. The flow directions shown for the cycles are arbitrary — any one of them can be reversed, independently of the others.

Let us begin by assuming that there are two limit cycles in the region R. Then they must both enclose the hole and one of them must enclose the other, as in the example of the limit cycles Γ_1 and Γ_2 in the right panel of figure 8.1. Consider now the **region enclosed between these two limit cycles (call it Ω, with $\partial\Omega = \Gamma_1 + \Gamma_2$ its boundary),** and let us write the equations the limit cycles satisfy as

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}),$$

where $\mathbf{x} = (x,\, y)$ and $\mathbf{F} = (f,\, g)$ are 2-vector valued. What we know (these are the hypothesis of Dulac's criterion) is that:

There exists a continuously differentiable, real valued function $h = h(\mathbf{x})$ **such that** $\nabla \cdot (h\,\mathbf{F})$ **has one sign throughout** R**.** Green's theorem then tells us that:

$$0 \neq \int_\Omega \nabla \cdot (h\,\mathbf{F})\,dA = \oint_{\partial\Omega} h\,\mathbf{F}\cdot\mathbf{n}\,dS = 0,$$

where $\mathbf{n}$ is the unit normal to the boundary $\partial\Omega$, the first inequality follows from Dulac's hypothesis above, and the last equality follows from the assumption that the boundary of Ω is made up by limit cycles. This is a **contradiction, thus we cannot have two limit cycles.**

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9 Problem 07.03.10 - Strogatz (Existence/non-existence of a limit cycle)

9.1 Statement for problem 07.03.10

Consider the two dimensional system

$$\frac{d\mathbf{x}}{dt} = A\,\mathbf{x} - r^2\,\mathbf{x}, \tag{9.1}$$

where $\boldsymbol{r} = \|\mathbf{x}\| = \sqrt{\boldsymbol{x_1^2 + x_2^2}}$, A is a 2×2 constant real matrix with complex eigenvalues $\alpha \pm i\,\beta$, and $\beta \neq 0$. Show that there exists at least one periodic orbit if $\alpha > 0$, and that there are none for $\alpha < 0$.

Hint. Think before you do anything. This is quite easy if you go at it the right way, but if not ...
Think of $\dot r$ for r large — what can you say about it?
Useful vector stuff: for any matrix and vector, $\|A\,\mathbf{x}\| \le \|A\|\,r$, where $\|A\| \ge 0$ is the norm of the matrix.
Useful vector stuff: $|\mathbf{a}\cdot\mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|$, *where* **the dot $\cdot$ indicates the scalar product.**

9.2 Answer for problem 07.03.10

Clearly the origin is the only critical point of the system, since any nonzero critical point would require that the corresponding r^2 be an eigenvalue of A.

A. Case $\boldsymbol{\alpha > 0}$. Then

$$r\,\frac{dr}{dt} = \mathbf{x}\cdot(A\,\mathbf{x}) - r^4. \tag{9.2}$$

Since $|\mathbf{x}\cdot(A\,\mathbf{x})| \le \|A\|\,r^2$, it follows that

$$\frac{dr}{dt} < 0 \qquad \text{for} \quad r > \sqrt{\|A\|}. \tag{9.3}$$

Thus the orbits cannot approach ∞ as $t \to \infty$. Further: they cannot approach the (single) critical point at the origin — because it is an unstable spiral. Hence, from the *Poincaré Bendixon theorem,* **there must be at least one periodic orbit.**

B. Case $\boldsymbol{\alpha < 0}$. A simple calculation shows that

$$\nabla\cdot\left(A\,\mathbf{x} - r^2\,\mathbf{x}\right) = \mathsf{Trace}(A) - 4\,r^2 = 2\,\alpha - 4\,r^2 < 0. \tag{9.4}$$

Thus, from *Dulac's criteria,* **there are no periodic orbits.**

#1. Extra task to the reader. Note that the result in **B** does not depend on $\beta \neq 0$. In fact, it only needs the trace of A to be non positive to apply.[3] On the other hand, **A** depends crucially on $\beta \neq 0$. **Question:** Does the system

[3] Dulac's criteria, as stated in the book, requires a strict inequality $<$ in (9.4). However, violation of this at a single point does not invalidate the proof — which relies on an integral of the left hand side being nonzero over.

have any periodic orbits when the eigenvalues of A are real, and at least one of them is positive? In particular: describe the phase plane for the system when A corresponds to a star, with positive eigenvalue α.

#2. Challenge to the reader. When $\boldsymbol{\alpha > 0}$ (the case in item **A**) the periodic orbit is, in fact, unique — hence a globally attracting limit cycle. **Can you prove this?**

Hint. The way to do it is by using "Dulac's criterion on an annulus" (see problem 7.2.17), the annulus being the region $0 < r$. *A function* g *such that* $\operatorname{div}(g\,\dot{\mathbf{x}}) > 0$ *can be obtained as a function of a single variable, but this variable is not something simple as* r. *In fact, let* $\mathbf{u}$ *and* $\mathbf{v}$ *be unit vectors along the principal directions of the ellipses associated*[4] *with* A, *and write any vector as* $\mathbf{x} = a\,\mathbf{u} + b\,\mathbf{v}$. *Then* $g = g(R)$, *where* $R = \sqrt{a^2 + b^2}$.

10 Multiple scales and limit cycles #01

10.1 Statement: Multiple scales and limit cycles #01

Consider the equation

$$\frac{d^2x}{dt^2} - \epsilon\,\cos x\,\frac{dx}{dt} + \frac{1}{\sqrt{\epsilon}}\,\sin(\sqrt{\epsilon}\,x) = 0, \quad \text{where} \quad \boldsymbol{0 < \epsilon \ll 1.} \tag{10.1}$$

Use a multiple scales analysis to calculate the frequency, stability and amplitude of any limit cycle (the frequency up to the first correction beyond linear and the amplitude up to leading order).

Hint. (i) Using the method in the "Weakly Nonlinear Oscillators" notes, you will find that the leading order approximation has the form $\boldsymbol{x_0 = A(\tau)\,e^{i\,t} + \text{c.c.}}$, *and then you will find an equation for* $\boldsymbol{A}$ *by suppressing resonances. As in the notes, substitute* $\boldsymbol{A = \frac{1}{2}\,\rho\,e^{i\,\phi}}$ *in this equation, to reduce it to two real valued equations for* $\boldsymbol{\rho}$ *and* $\boldsymbol{\phi}$.

(ii) To write the equation for $\boldsymbol{\rho}$ *in a compact way, use the equality* $\boldsymbol{\pi\,J_1(\rho) = \int_0^{2\pi} \sin(\rho\,\cos(s))\,\cos(s)\,ds}$, *where* $\boldsymbol{J_1}$ *is the index one Bessel function of the first kind. Because* $\boldsymbol{J_1}$ *is well studied and understood, this will allow you to conduct the required analysis.*

10.2 Answer: Multiple scales and limit cycles #01

This problem is very similar to the van der Pol equation problem in the *"Weakly Nonlinear Oscillators"* notes. Thus, consider a two times expansion of the form

$$x = x_0(t,\,\tau) + \epsilon\,x_1(t,\,\tau) + \epsilon^2\,x_2(t,\,\tau) + \dots \tag{10.2}$$

where $\tau = \epsilon\,t$. Substituting this expansion into equation (10.1) and collecting equal powers of ϵ we find:

At $O(1)$:

$$\frac{\partial^2 x_0}{\partial\,t^2} + x_0 = 0 \quad \Longrightarrow \quad x_0 = A(\tau)\,e^{i\,t} + \mathbf{c.c.}, \tag{10.3}$$

where **c.c.** denotes the complex conjugate and $A = A(\tau)$ is a function to be determined at higher order.

At $O(\epsilon)$:

$$\begin{aligned} \frac{\partial^2 x_1}{\partial t^2} + x_1 &= -2\frac{\partial^2 x_0}{\partial t\,\partial\tau} + cos(x_0)\,\frac{\partial x_0}{\partial t} + \frac{1}{6}\,x_0^3 \\ &= \left(-2\,i\,\frac{dA}{d\tau}\,e^{i\,t} + \frac{1}{2}\,|A|^2\,A\,e^{i\,t} + \frac{1}{6}\,A^3\,e^{3\,i\,t}\right) + \mathbf{c.c.} + \frac{\partial}{\partial t}\sin\left(A\,e^{i\,t} + \mathbf{c.c.}\right). \end{aligned} \tag{10.4}$$

This has the form

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = F(t,\tau), \tag{10.5}$$

[4] Take an eigenvector for A. Then $\mathbf{u}$ and $\mathbf{v}$ are unit vectors along the real and imaginary parts of the eigenvector.

where the forcing $F = F(t,\,\tau)$ is a real valued function of period 2π in t. In terms of the Fourier series

$$F = \sum_{n=-\infty}^{\infty} F_n(\tau)\, e^{i\,n\,t},$$

we see that we only have to worry about the forcing given by the Fourier terms with $n = \pm 1$, since all the others cause a periodic response in x_1. Since F_{-n} is the complex conjugate of F_n, the elimination of secular terms in x_1 leads to the condition:

$$0 = F_1(\tau) = \frac{1}{2\,\pi}\int_0^{2\pi} F(t,\,\tau)\, e^{-i\,t} dt = -2\,i\left\{\frac{dA}{d\tau} + \frac{i}{4}|A|^2 A - \frac{1}{4\pi}\int_0^{2\pi} \sin(Ae^{i\,t} + \mathbf{c.c.})e^{-i\,t} dt\right\}. \tag{10.6}$$

Introduce now $\boldsymbol{A = \frac{1}{2}\rho e^{i\phi}}$, where both $\rho = \rho(\tau) > 0$ and $\phi = \phi(\tau)$ are real valued. Then

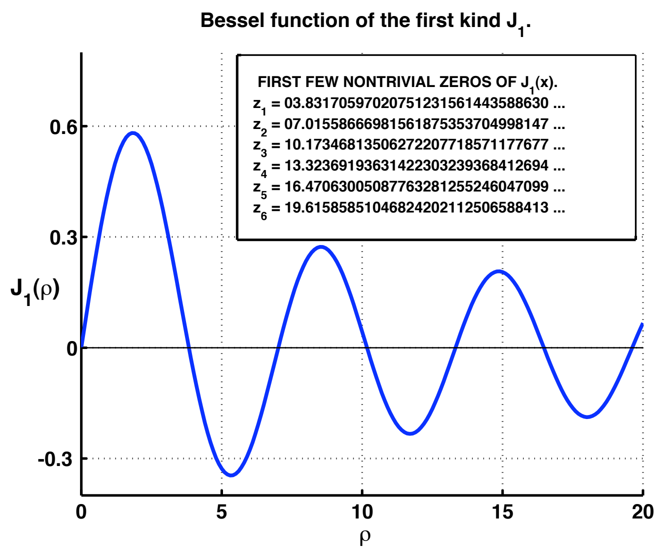

Figure 10.1: Bessel function of the first kind J_1

$$\begin{aligned}\int_0^{2\pi} \sin(Ae^{it} + \mathbf{c.c.})e^{-it}dt &= \int_0^{2\pi} \sin(\rho\cos(t+\phi))e^{-it}dt \\ &= 2e^{i\phi}\int_0^{\pi} \sin(\rho\cos(s))\cos(s)ds \;=\; 2\pi e^{i\phi}J_1(\rho),\end{aligned}$$

where J_1 is the index one Bessel function of the first kind. Thus equation (10.6) becomes [5]

$$\boldsymbol{\frac{d\rho}{d\tau} = J_1(\rho)} \qquad \text{and} \qquad \boldsymbol{\frac{d\phi}{d\tau} = -\frac{1}{16}\,\rho^2.} \tag{10.7}$$

If $z_0 = 0 < z_1 < z_2, \dots$ are the zeros of J_1 (see figure 10.1), then $z_1,\, z_3,\, z_5, \dots$ correspond to *stable limit cycles,* $z_2,\, z_4, \dots$ correspond to unstable limit cycles, and the origin is an *unstable spiral point.*

Putting it all together we see that **approximate expressions for the limit cycles** are given by

$$\boldsymbol{x \sim z_n\,\cos(\omega\,(t - t_0)) + O(\epsilon),} \qquad \text{where} \qquad \boldsymbol{\omega = 1 - \frac{1}{16}\,z_n^2\,\epsilon + O(\epsilon^2)} \tag{10.8}$$

[5] Note that the right hand side for the phase ϕ equation is due exclusively to the influence of the term $\boldsymbol{\epsilon^{-1}\sin(\epsilon x)}$ in (10.1), while the forcing on the radial evolution is due exclusively to the influence of the term involving $\dot{\boldsymbol{x}}$.

and $\boldsymbol{t_0}$ is an arbitrary constant. Of course, here $\boldsymbol{n}$ **cannot be too large,** since the expansion assumes that the $\boldsymbol{x_j}$'s in (10.2) are $\boldsymbol{O(1)}$ quantities.

THE END.