Answers to P-Set # 03, 18.385j/2.036j
MIT (Fall 2020)

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1.1 Statement: Critical points in the phase plane and "energies"

Both in the lectures and the book (see example 6.5.1) this claim was made:

$$\textbf{Conservative systems can have neither attracting, nor repelling, fixed points.} \tag{1.1}$$

Consider now the following (linear) systems, with an isolated fixed point at the origin:

$$\left.\begin{aligned} \frac{dx}{dt} &= -x, \\ \frac{dy}{dt} &= a\,y, \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} \frac{dx}{dt} &= -b\,x - y, \\ \frac{dy}{dt} &= -b\,y + x, \end{aligned}\right\} \tag{1.3}$$

where $a \neq 0$, and b are constants.

For the system in (1.2), define E_a by

$$E_a = |x|^a\,y. \tag{1.4}$$

Then, it is easy to check that:

$$\frac{d\boldsymbol{E_a}}{d\boldsymbol{t}} = 0 \quad \textbf{along solutions of (1.2).} \tag{1.5}$$

Hence E_a is a conserved quantity. Now, for $a > 0$ the origin in (1.2) is a saddle, so that this last fact is not in contradiction with (1.1). However:

$$\textbf{(1.5) applies even when } a < 0 \textbf{ and the origin is an } \underline{\textbf{attracting}} \textbf{ node!} \tag{1.6}$$

Similarly, let:

$$E_b = 2\,b\,\arctan\left(\frac{y}{x}\right) + \ln(x^2 + y^2) = 2\,b\,\theta + \ln(r^2), \tag{1.7}$$

where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then, again, it is easy to check that:

$$\begin{gathered} \frac{d\boldsymbol{E_b}}{d\boldsymbol{t}} = 0 \textbf{ for the solutions of (1.3),} \\ \textbf{even when } b \neq 0 \textbf{ and the origin is an } \underline{\textbf{attracting/repelling}} \textbf{ spiral point.} \end{gathered} \tag{1.8}$$

Question: Do either of (1.6) or (1.8) contradict (1.1)? Explain.

1.2 Answer: Critical points in the phase plane and "energies"

There is no contradiction between either (1.6) or (1.8) and (1.1). Conserved quantities must be **continuous**, non-constant (on every open set) functions for (1.1) to apply — which is not true for either E_a or E_b under the conditions above. In particular, notice that the argument in example 6.5.1 in the book uses continuity at the critical point.

As a final point, observe that (for $b \neq 0$) E_b is not even single valued, since any multiple of 2π can be added to θ. This, however, is not an issue: for $b \neq 0$ one can define a single valued function (which is also conserved), via $E_b^* = \sin(\frac{1}{b}E_b)$. But E_b^* is still singular at the critical point.

2 Critical points with zero linear part

2.1 Statement: Critical points with zero linear part

Consider the following phase plane systems

$$\dot{x} = (x^2+y^2)\,y \quad \text{and} \quad \dot{y} = -(x^2+y^2)\,x, \tag{2.1}$$
$$\dot{x} = (x^2+y^2)\,(y-x/9) \quad \text{and} \quad \dot{y} = -(x^2+y^2)\,(x+y/9), \tag{2.2}$$
$$\dot{x} = -(x^2+y^2)\,x \quad \text{and} \quad \dot{y} = -(x^2+y^2)\,y/2, \tag{2.3}$$
$$\dot{x} = (x^2+y^2)\,x \quad \text{and} \quad \dot{y} = -(x^2+y^2)\,y. \tag{2.4}$$

In each of these systems the origin, $\mathcal{O}$, is the only critical point — with linearization matrix $A = 0$. Yet each of these systems has a phase portrait that is entirely analogous to that of a linear system (saddle, center, node, etc.). **Analyze the systems above, classify and sketch their phase plane portraits, and calculate the index for $\mathcal{O}$ in each case.**

Further questions. **(Q1)** For a stable node in a linear system, the solutions approach the critical point exponentially fast as $t \to \infty$. *If one of the systems above has a stable node at the origin, at what rate does $r = \sqrt{x^2+y^2}$ vanish as t grows?* **(Q2)** For a linear center, the orbital period is independent of the amplitude. *If one of the systems above has a center at the origin, how does the orbital period scale with amplitude?*

2.2 Answer: Critical points with zero linear part

Upon re-scaling time via $\boldsymbol{d\tau = (x^2+y^2)\,dt}$, the equations become

$$\frac{dx}{d\tau} = y \quad \text{and} \quad \frac{dy}{d\tau} = -x, \tag{2.5}$$
$$\frac{dx}{d\tau} = -x/9 + y \quad \text{and} \quad \frac{dy}{d\tau} = -x - y/9, \tag{2.6}$$
$$\frac{dx}{d\tau} = -x \quad \text{and} \quad \frac{dy}{d\tau} = -y/2, \tag{2.7}$$
$$\frac{dx}{d\tau} = x \quad \text{and} \quad \frac{dy}{d\tau} = -y, \tag{2.8}$$

respectively. These are the linear equations for a center, a stable spiral, a stable node, and a saddle. Hence the indexes are: 1, 1, 1, and -1, respectively. Their phase plane portraits can be found in figure 2.1.

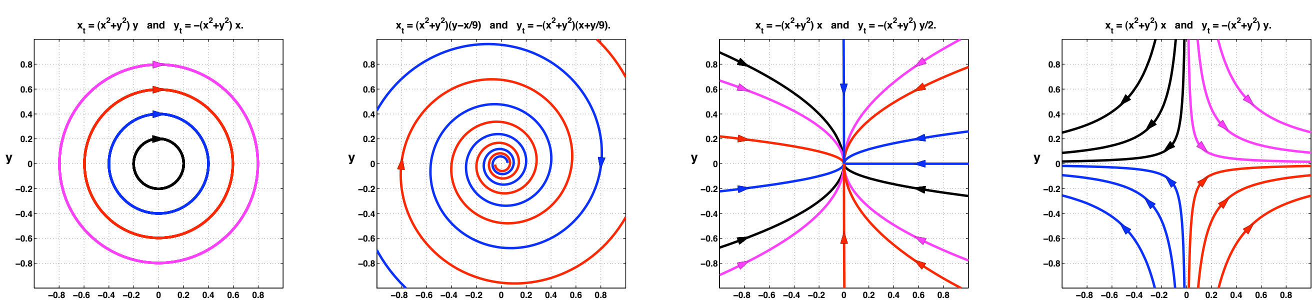

Figure 2.1: Phase plane portraits for the systems in (2.1 – 2.4) — ordered left to right.

(Q1) Consider the solutions to (2.7), which has a stable node at the origin. These solutions vanish, as $\tau \to \infty$, like $e^{-\mu\,\tau}$ ($\mu = 0.5$ or $\mu = 1$); i.e., $x^2+y^2 = O(e^{-2\,\mu\,\tau})$. Thus, from $dt/d\tau = (x^2+y^2)^{-1}$, we see that $t = O(e^{2\,\mu\,\tau})$ as $\boldsymbol{\tau \to \infty}$ — or $\boldsymbol{\tau \sim \frac{1}{2\,\mu}\,\ln t}$. It follows that $\boldsymbol{r}$ **vanishes algebraically as $\boldsymbol{t \to \infty}$, in fact like $\boldsymbol{1/\sqrt{t}}$.**

This argument applies to any stable node, or stable spiral, for that matter. In the particular case of (2.3), we note that, for solutions where y does not vanish, $|x| \ll |y|$ as $t \to \infty$ — see the third panel in figure 2.1. Thus the equations reduce to $\dot{y} \sim -\frac{1}{2}\, y^3$ and $\dot{x} \sim -y^2\, x$. Solving this shows that $y \sim 1/\sqrt{t}$ and $x \sim -1/t$, from which the conclusion follows. For solutions where $y \equiv 0$, $\dot{x} = -x^3$, and we obtain the same answer.

(Q2) In the case of a center, the relationship between the period in terms of τ — say ϕ, and the actual period in terms of t — say T, is:

$$T = \int_0^T dt = \int_0^\phi \frac{d\tau}{x^2 + y^2} \sim \frac{\phi}{A^2},$$

where A is the amplitude of the orbit [for example, the average between the largest and smallest values of r]. It follows that: **for centers the period scales like the inverse amplitude squared.**

Again, the argument above is generic. For the particular case of (2.1), we can write the exact solutions easily: $\boldsymbol{x = A\, \sin(A^2\, t + \theta_0)}$ and $\boldsymbol{y = A\, \cos(A^2\, t + \theta_0)}$, where A and θ_0 are constants.

3 Index for a center when linearized

3.1 Statement: Index for a center when linearized

Consider a phase plane system

$$\dot{x} = f(x,\, y) \quad \text{and} \quad \dot{y} = g(x,\, y), \tag{3.1}$$

where f and g are smooth functions of all of its arguments. Assume that:

1. The origin is a critical point. That is $f(0,\, 0) = g(0,\, 0) = 0$.
2. The origin is a center for the linearized system. That is: if A is the 2×2 matrix corresponding to the linearized system near the critical point at the origin, then A has two complex conjugate, pure imaginary and nonzero, eigenvalues. Equivalently, if $\tau = \text{tr}(A)$ and $\Delta = \det(A)$, then $\boldsymbol{0 < \Delta}$ and $\boldsymbol{\tau = 0}$.

Because $\Delta \neq 0$, the origin is an isolated critical point (inverse function theorem), and has an index associated with it. Let this index be $\mathcal{I}_0$. **Your task here is to calculate $\boldsymbol{\mathcal{I}_0}$.**

Note. Centers are structurally unstable. Hence *you cannot calculate their index by simply calculating the index for the linearized system,* you have to do a slightly more sophisticated calculation. Interestingly, even though the actual phase plane portrait (when a center occurs for the linearized system) cannot be ascertained from the linearized system alone (structural instability), you need no nonlinear information to calculate the index!

Hint:

3. Write the system (3.1) in vector form

$$\dot{Y} = F(Y), \qquad \text{where } \; Y = (x,\, y)^T \; \text{ and } \; F = (f,\, g)^T. \tag{3.2}$$

where $Y = (x,\, y)^T$ and $F = (f,\, g)^T$.

4. Assume that the matrix A has the form
$$A = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, \tag{3.3}$$
where $\mu > 0$ and $\lambda = \pm i\, \mu$ are the eigenvalues of A. There is no loss of generality here: any matrix with two nonzero, conjugate and purely imaginary eigenvalues, can be reduced to this form by an appropriate choice of coordinates.

5. Consider now the one parameter family of systems

$$\dot{Y} = F(Y) + \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y. \tag{3.4}$$

For these systems the origin is a critical point, with the linearization matrix

$$B = \begin{pmatrix} \epsilon & \mu \\ -\mu & \epsilon \end{pmatrix}. \qquad \text{In particular } \det(B) = \epsilon^2 + \mu^2 > 0, \tag{3.5}$$

so that the origin is an isolated critical point (inverse function theorem).

Let now $\mathcal{I} = \mathcal{I}(\epsilon)$ be the index of the critical point at the origin for (3.4). Calculate $\mathcal{I}(\epsilon)$ for $\epsilon \neq 0$, and use the properties of the index to get $\mathcal{I}(0) = \mathcal{I}_0$.

3.2 Answer: Index for a center when linearized

The eigenvalues for B are

$$\lambda = \epsilon \pm i\,\mu, \tag{3.6}$$

It follows that, for $\epsilon \neq 0$, the origin in equation (3.4) is a spiral. Hence $\mathcal{I}(\epsilon) = 1$ for $\epsilon \neq 0$. However, (3.4) depends continuously on ϵ. Thus $\mathcal{I}(\epsilon)$ is a continuous function. **We conclude that $\boldsymbol{\mathcal{I}(0) = \mathcal{I}_0 = 1}$.**

4 Index for a critical point with zero determinant

4.1 Statement: Index for a critical point with zero determinant

Consider a phase plane system

$$\dot{x} = f(x,\, y) \quad \text{and} \quad \dot{y} = g(x,\, y), \tag{4.1}$$

where f and g are smooth functions of all of its arguments. Assume that:

1. The origin $\mathcal{O}$ is an isolated critical point. That is $f(0,\,0) = g(0,\,0) = 0$, and there are no solutions to $f(x,\,y) = g(x,\,y) = 0$ with $0 < x^2 + y^2 < \epsilon$ — for some ϵ.

2. Let A be the 2×2 matrix corresponding to the linearized system near $\mathcal{O}$, with $\tau = \mathrm{tr}(A)$ and $\Delta = \det(A)$. Suppose that $\boldsymbol{\Delta = 0}$ and $\boldsymbol{\tau > 0}$ — so that one eigenvalue of A vanishes, and the other equals τ.

This is a structurally unstable situation, in particular: the index for $\mathcal{O}$ is not determined at all by the linearized equations. **Construct examples of the above situation where:**

A. $\mathcal{I} = \text{index}(\mathcal{O}) = \;\;1.$

B. $\mathcal{I} = \text{index}(\mathcal{O}) = -1.$

C. $\mathcal{I} = \text{index}(\mathcal{O}) = \;\;0.$

Sketch the phase plane portraits for the systems that you construct.

Hints. Consider the linear system $\dot{Y} = AY$, and then add a nonlinear correction which:

For part **A**. Makes $\mathcal{O}$ into a (nonlinear) node.

For part **B**. Makes $\mathcal{O}$ into a (nonlinear) saddle.

For part **C**. Makes $\mathcal{O}$ into a (nonlinear) saddle on one side, and a (nonlinear) node on the other.

4.2 Answer: Index for a critical point with zero determinant

We take A of the form
$$A = \begin{pmatrix} 0 & 0 \\ 0 & \tau \end{pmatrix}, \tag{4.2}$$
and then follow the strategy in the hints:

A. Let
$$\dot{x} = x^3 \quad \text{and} \quad \dot{y} = \tau\, y. \tag{4.3}$$
This system has a (nonlinear) node at the origin. It, effectively, behaves as a "linear" system with two positive, and non-equal, eigenvalues: $\lambda = \tau$ and $\lambda = x^2$. Hence $\boldsymbol{\mathcal{I} = 1}$. The phase plane portrait for this system can be found on the left panel of figure 4.1.

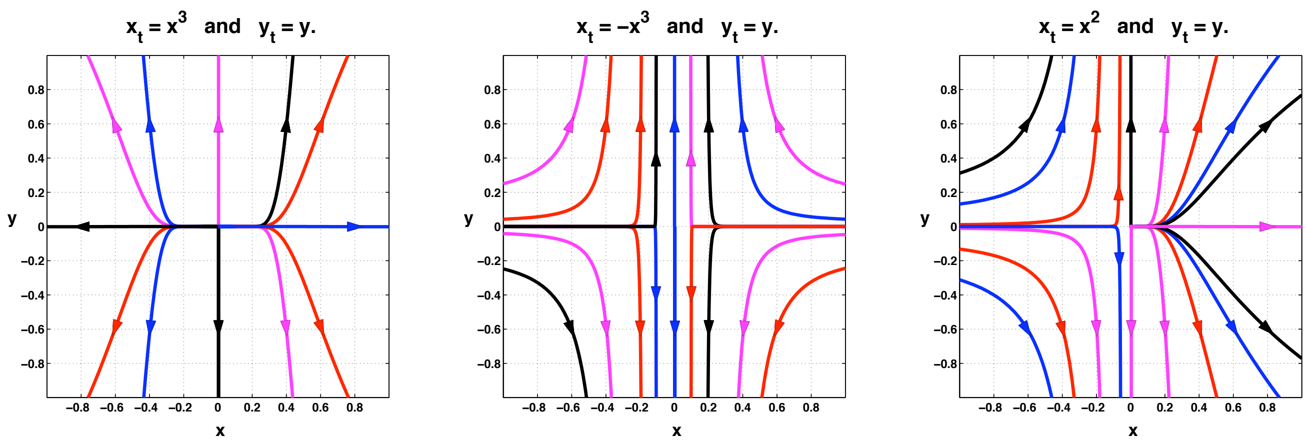

Figure 4.1: Phase plane portraits for the systems in (4.3 – 4.5) — ordered left to right.

B. Take
$$\dot{x} = -x^3 \quad \text{and} \quad \dot{y} = \tau\, y. \tag{4.4}$$
This system has a (nonlinear) saddle at the origin. It, effectively, behaves as a "linear" system with eigenvalues: $\lambda = \tau > 0$ and $\lambda = -x^2 < 0$. Hence $\boldsymbol{\mathcal{I} = -1}$. The phase plane portrait for this system can be found on the middle panel of figure 4.1.

C. Take
$$\dot{x} = x^2 \quad \text{and} \quad \dot{y} = \tau\, y. \tag{4.5}$$
This system, effectively, behaves as a "linear" system with eigenvalues: $\lambda = \tau > 0$ and $\lambda = x$. Hence, for $x > 0$ node-like behavior occurs, while for $x < 0$ saddle-like behavior occurs. It follows that $\boldsymbol{\mathcal{I} = 0}$. The phase plane portrait for this system can be found on the right panel of figure 4.1.

5 Index theory - interpolating from saddles to nodes #2

5.1 Statement: Index theory - interpolating from saddles to nodes #2

Consider a one parameter family of phase plane systems
$$\dot{x} = f(x,\, y,\, r) \quad \text{and} \quad \dot{y} = g(x,\, y,\, r), \tag{5.1}$$
where f and g are smooth functions of all of its arguments — including the parameter r. Assume that:

1. The origin is a critical point for all values of r. That is $f(0,\,0,\,r) = g(0,\,0,\,r) = 0$.

2. For $\boldsymbol{r = 0}$ the origin is an isolated critical point. In fact, a saddle.

3. For $\boldsymbol{r = 1}$ the origin is an isolated critical point. In fact, a node.

Show that there is at least one value $0 < R < 1$, such that: for $r = R$ the origin is not isolated critical point.

Hint. Let $\mathcal{I} = \mathcal{I}(r)$ be the index of the critical point at the origin for (5.1), for any r for which it is defined. Also note that $\mathcal{I}(0) = -1$ and $\mathcal{I}(1) = 1$. Use now the properties of the index.

5.2 Answer: Index theory - interpolating from saddles to nodes #2

A key point to notice here is that: **the index is defined for any isolated critical point, and only for those.** One needs to be able to enclose the critical point within a closed curve Γ such that:

3. There are no critical points along Γ.

2. The critical point in question is the only critical point within Γ.

In addition, the index is continuous[1] and integer valued. Hence $\mathcal{I}$ *should be a constant on any interval where it is defined.* Yet $\mathcal{I}(0) = -1$, while $\mathcal{I}(1) = 1$. It follows that *there must be at least one value $0 < R < 1$, such that for $r = R$ the index $\mathcal{I}(R)$ is not defined.* Hence **for $r = R$ the origin cannot be an isolated critical point.**

6 Liapunov Function # 01

6.1 Statement: Liapunov Function # 01

Show that the system
$$\frac{dx}{dt} = -x + 2\,y^3 - 2\,y^4 \quad \text{and} \quad \frac{dy}{dt} = -x - y + x\,y, \tag{6.1}$$
has no periodic solutions.

Hint. Find a Liapunov function. Try the form $L = x^m + a\,y^n$.

6.2 Answer: Liapunov Function # 01

Let $L = \frac{1}{2}\,(x^2 + y^4)$. Then
$$\frac{dL}{dt} = -x^2 - 2\,y^4 \quad \Longrightarrow \quad \frac{dL}{dt} < 0, \quad \text{unless} \;\; x = y = 0. \tag{6.2}$$
Thus L is a Liapunov function, and all orbits approach the single (global) minimum of L at the origin.

7 Example of a reversible system that is not conservative

7.1 Statement: Example of a reversible system that is not conservative

Give an example of a reversible system that is not conservative.
Hint. *Remember that a phase plane system*
$$\dot{x} = f(x,\,y) \quad \text{and} \quad \dot{y} = g(x,\,y), \tag{7.1}$$
is reversible if, for example, f is odd and g is even in y — that is: $f(x,\,-y) = -f(x,\,y)$ and $g(x,\,-y) = g(x,\,y)$. In this case the change $t \to -t$ and $y \to -y$ leaves the system invariant.

[1] The index is continuous with respect to parameters on which the system depends continuously, as (5.1) does on r.

In addition, we know that conservative systems cannot have sinks or sources. Now, ask yourself: what systems have exactly the opposite property [almost every critical point is either a source or a sink]. Then produce a system of this kind, with f odd and g even.

7.2 Answer: Example of a reversible system that is not conservative

Consider a **gradient system:** $\dot{\boldsymbol{x}} = -\boldsymbol{V_x}$ and $\dot{\boldsymbol{y}} = -\boldsymbol{V_y}$, where $\boldsymbol{V} = \boldsymbol{V(x,\, y)}$ is some given function. For systems of this type, any local minimum of V is a sink, and any local maximum is a source. Now **take $\boldsymbol{V}$ odd in $\boldsymbol{y}$; then the system is also reversible.** Hence any gradient system where V is odd in y, and where V has either a local maximum or a local minimum, provides the required example. One is $\boldsymbol{V = y\,(1-y^2)/(1+x^2)}$, which has a local minimum at $(x,\, y) = (0,\, -1/\sqrt{3})$, and a local maximum at $(x,\, y) = (0,\, 1/\sqrt{3})$.

8 Three limit cycles and enclosed critical points

8.1 Statement: Three limit cycles and enclosed critical points

Consider a phase plane system such that, in some region $\boldsymbol{R}$ of the phase plane it has two disjoint limit cycles, both enclosed by a third one — the situation is illustrated in figure 8.1. **For this to be possible: what is the minimum number of critical points that the system needs to have in $\boldsymbol{R}$?**

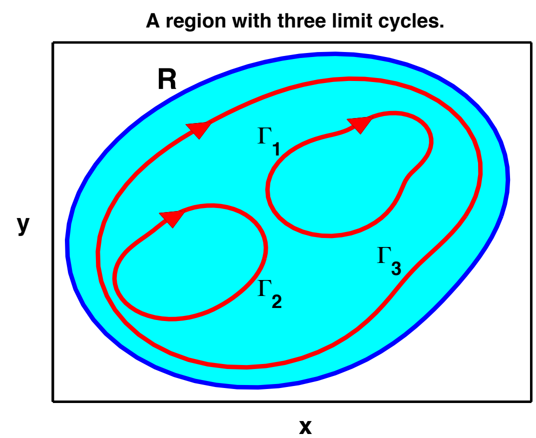

Figure 8.1: Two disjoint limit cycles, both enclosed by a third one, in some region $\boldsymbol{R}$ of the phase plane. What is the minimum number of critical points needed for this to happen?

8.2 Answer: Three limit cycles and enclosed critical points

From index theory we know that each of the closed orbits has index one. Hence there must be, at least, one critical point inside each of Γ_1 and Γ_2 (see figure 8.1), and the sum of the indexes of the critical points inside each of these orbits must be one. Furthermore: in the region outside Γ_1 and Γ_2, but inside Γ_3, there must be at least one critical point, and the sum of the indexes of the critical points in this region must be minus one. Thus, the answer is **a minimum of three critical points are needed.** For example: a spiral point inside each of Γ_1 and Γ_2, and a saddle point inside Γ_3, but outside Γ_1 and Γ_2.

THE END.