Answers to P-Set # 02, 18.385j/2.036j
MIT (Fall 2020)

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1.1 Statement: Irreversible switch using saddle node and transcritical bifurcations

Imagine a system[1] with a controlling parameter r, and with (at most) two distinct stable equilibrium states: $x_1 = x_1(r)$ and $x_2 = x_2(r)$. In particular, such that *infinity is unstable* — that is: for every solution $x = x(t)$ there exists a constant $M > 0$ such that $|x| < M$ for t large enough. Furthermore:

A. There is a value $r = r_s =$ **switch value** such that: for $r > r_s$ both states exist and are stable — so that the system can be in either one of them.

B. For $r < r_s$ only the state x_1 exists and it is stable.

C. Both $x_1(r)$ and $x_2(r)$ are continuous functions of r (though, maybe, not smooth), and $|x_1(r) - x_2(r)|$ is bounded away from zero.

Such a system, if started in the state x_2 for $r > r_s$, remains in x_2 for as long as r varies (slowly enough) in the range $r > r_s$. Once r crosses below the threshold r_s, the system switches to x_1, and remains there for all values of r. A switch back to x_2 is not produced by slow variations in r. The condition in item **C** is important, for otherwise small perturbations could produce an "accidental" switch if x_1 and x_2 get very close.

Remark 1.1 *A "standard" (reversible) switch [e.g.: a thermostat], operates using hysteresis. For such systems there are two switching values $r_1 < r_2$, with only x_2 stable for $r > r_2$, only x_1 stable for $r < r_1$, and both states stable for $r_1 \leq r \leq r_2$. Then the system jumps from x_2 to x_1 as r is lowered below r_1, and goes back to x_2 as r is raised above r_2.* ♣

Construct an irreversible switch, using a 1-D system of the form

$$\frac{dx}{dt} = f(x, r), \tag{1.1}$$

with the behavior caused by two bifurcations: a trans-critical and a saddle node *(no other bifurcations should occur!)* **Then draw the bifurcation diagram.**

Hint: It is very easy to construct an explicit example in which f in equation (1.1) is a cubic polynomial in x, and it is linear in the parameter r.

[1] A "switch".

Remark 1.2 (Switch uniqueness). *Even for a 1-D system such as the one in (1.1), there is an infinite number of possible bifurcation diagrams that yield a switch, with various types of bifurcations involved.*[2] *However, if the restriction that there should be only two bifurcations (one saddle-node and one transcritical) is imposed, then there are only two possible topologies for the switch bifurcation diagram. This problem asks you to produce an example of one such switch.* ♣

1.2 Answer: Irreversible switch using saddle node and transcritical bifurcation

Take $f = (r-(x-1)^2)x$ in (1.1), so that the equation becomes

$$\frac{dx}{dt} = (r-(x-1)^2)x. \tag{1.2}$$

The bifurcation diagram for this equation is easily computed — see figure 1.1. It should be clear that the switch

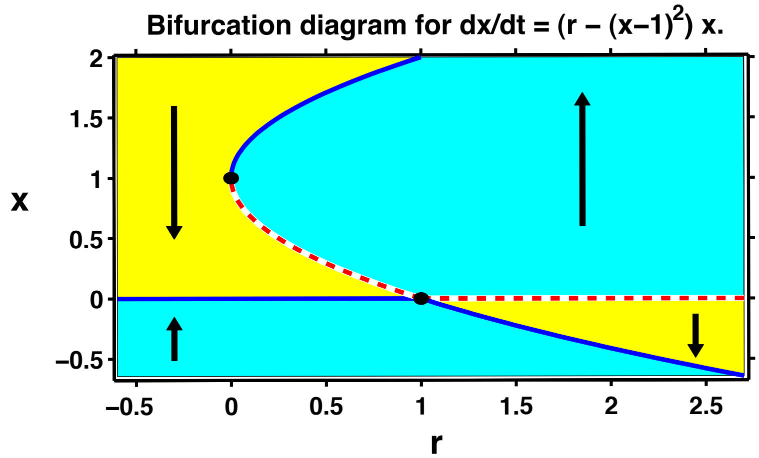

Figure 1.1: Bifurcation diagram for the irreversible switch given by equation (1.2).

value is $r_s = 0$, with the (stable) equilibrium states given by: **(1)** $\boldsymbol{x_2 = 1+\sqrt{r}}$ for $r \geq 0$, and **(2)** $\boldsymbol{x_1 = 0}$ if $r \leq 1$ and $\boldsymbol{x_1 = 1-\sqrt{r}}$ if $r \geq 1$.

Interestingly, the bifurcation diagram in figure 1.1 is the one that occurs when a perturbation breaks the symmetry of a system with a soft pitchfork bifurcation (but without destroying the continuity of the middle fork).[‡] An example of a physical system that behaves like the "switch" in figure 1.1 is a measuring tape subject to longitudinal pressure, though you would not think of it as a "switch". In fact, for this system a "switch" to the state x_2 is bad, as it carries the risk of permanently deforming the tape. ♣

[‡] See the problems "Perturbed pitchfork, with root preserved".

Remark 1.3 *Note that x_1 and x_2 are both continuous functions of r, but neither is smooth everywhere: $x_1'(r)$ is discontinuous at $r = 1$, and $x_2'(r) \to \infty$ as $r \downarrow 0$.* ♣

Figure 1.2 shows another possible bifurcation diagram for an irreversible switch involving a saddle node and a single transcritical bifurcation. *Any 1-D bifurcation diagram for an irreversible switch involving a saddle node and a single transcritical bifurcation is topologically equivalent to one of the diagrams in figures* 1.1 *or* 1.2 — shown in problem: *Irreversible switches; classification.*

[2] This is the subject of another problem: *Irreversible switches; classification.*

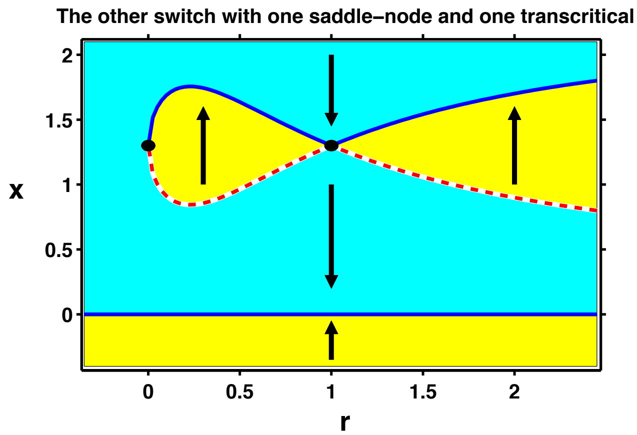

Figure 1.2: The alternative bifurcation diagram for an irreversible switch using a saddle-node and a transcritical bifurcation. The equation for this plot is:
$\dot{x} = (x - 1.3)^2 - r(r-1)^2/(0.7 + r^{1.5})^2$.

Remark 1.4 ***Best and worse switches.*** *Note that both the switch in figure* 1.1 *and the switch in figure* 1.2 *have potential problems. For the switch in figure* 1.1*, when r is close to one, a jump back from x_1 to x_2 is possible with very small perturbations —* switch failure*. In the same fashion, the switch in figure* 1.2 can fail *for r close to one, with the system jumping from x_2 to x_1 where it is not supposed to. The* best type of switch *that a system like (1.1) can produce is one where the unstable branch of solutions separating x_1 from x_2 remains as far away from x_1 and x_2 as possible (of course, it has to join x_2 for $r = r_s$). Thus the best switch has just a saddle-node bifurcation, and no other.*

2 Perturbed pitchfork, with root preserved (bifurcation diagram)

2.1 Statement: Perturbed pitchfork, with root preserved (bifurcation diagram)

Consider the structural stability for a (soft) pitchfork bifurcation, with the restriction that the "main" solution branch is preserved across the bifurcation. Specifically, consider the situation where:

$$\frac{dx}{dt} = g(x,\, r) \quad (g \text{ odd in } x), \tag{2.1}$$

has a (soft) pitchfork bifurcation at $(x,\, r) = (0,\, 0)$. Assume that the problem depends on a hidden parameter h — i.e. let $\boldsymbol{g(x,\, r) = f(x,\, r,\, h)\big|_{h=0}}$, where you only know that h is small (but it may not be zero). Assume also that you know that $\boldsymbol{f(0,\, r,\, h) = 0}$, though f may not be odd for $h \neq 0$. Provided that f is reasonably smooth, and f is generic, it can be shown that the canonical equation[3] describing this situation is

$$\frac{dx}{dt} = r\,x + h\,x^2 - x^3. \tag{2.2}$$

Tasks: Assume $h \neq 0$ small (say, $h = 0.05$), and **draw the bifurcation diagram** for (2.2), including the flow lines — recall that the bifurcation diagram is, basically, all the phase portraits (one for each r) stacked in one single 2-D plot. *What happens to the pitchfork?* Furthermore: *estimate the level of noise (in x) under which the distinction between the pitchfork and the new behavior will be hidden* — do this in terms of h.

[3] That is, near the bifurcation, the full problem can be mapped into equation (2.2).

2.2 Answer: Perturbed pitchfork, with root preserved (bifurcation diagram)

The **critical points for (2.2) are given by $x = 0$ for all values of $(r,\, h)$**, and

$$x_u = \frac{h}{2} + \sqrt{r + \frac{h^2}{4}}, \quad x_d = \frac{h}{2} - \sqrt{r + \frac{h^2}{4}}, \quad \text{for} \quad r \geq -\frac{h^2}{4}. \tag{2.3}$$

Alternatively, instead of (2.3), we can write

$$r = x^2 - h\,x = (x - \tfrac{1}{2}\,h)^2 - \tfrac{1}{4}\,h^2, \tag{2.4}$$

which **parameterizes the nonzero critical points by giving r as a function of x.** Before drawing the bifurcation diagram, we notice that we can **scale-out h from equation (2.2)** by the transformation

$$\boldsymbol{x = h\,X}, \quad \boldsymbol{r = h^2\,R}, \quad \text{and} \quad \boldsymbol{t = T/h^2}, \tag{2.5}$$

as long as $\boldsymbol{h \neq 0}$. This reduces (2.2) to

$$\frac{dX}{dT} = R\,X + X^2 - X^3. \tag{2.6}$$

The bifurcation diagram for this equation can be found in figure 2.1. Note that the pitchfork is broken into a transcritical and a saddle node (separated by a distance which is $O(h)$ in x — this follows from the scaling in (2.5).

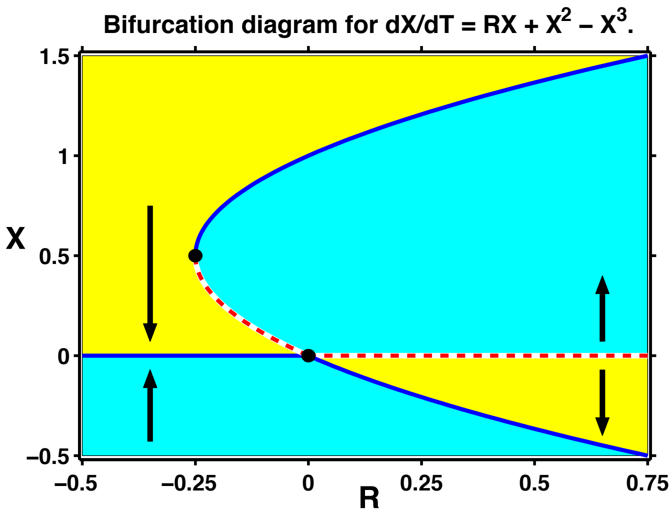

Figure 2.1: Scaled perturbation diagram for a perturbed pitchfork, with perturbation preserving the "main" solution. The canonical equation for the situation is (2.2), which (for $h \neq 0$) can be transformed into: $dX/dT = R\,X + X^2 - X^3$. The diagram on the left corresponds to this equation. The stable branches of solutions are indicated by solid blue lines, while the unstable ones are in dashed red. The black dots indicate the location of the bifurcations.

Remark *The scaling in (2.5) indicates that the changes to the pitchfork bifurcation diagram occur within a neighborhood of the critical point at x, whose size is $O(h)$. Outside this neighborhood the diagram looks the same as that of the pitchfork.* It follows that if the system is noisy, with a level of noise above $O(h)$, it becomes impossible to detect the difference between the diagram in figure 2.1, and that of the unperturbed system. ♣

3 Perturbed pitchfork, with root preserved (canonical form)

3.1 Statement: Perturbed pitchfork, with root preserved (canonical form)

Consider the structural stability for a (soft) pitchfork bifurcation, with the restriction that the "main" solution branch is preserved across the bifurcation. Specifically, consider the situation where:

$$\dot{y} = g(y,\, \lambda) \quad (g \text{ odd in } y), \tag{3.1}$$

has a (soft) pitchfork bifurcation at $(y,\,\lambda) = (0,\,0)$. Assume that the problem depends on a hidden parameter ρ — i.e. let $\boldsymbol{g(y,\,\lambda) = f(y,\,\lambda,\,\rho)\big|_{\rho=0}}$, where you know that ρ is small (but it may not be zero). Assume also that you know that $\boldsymbol{f(0,\,\lambda,\,\rho) = 0}$, though f may not be odd for $\rho \neq 0$. Provided that f is reasonably smooth, and f is generic, it can be shown that the canonical equation[4] describing this situation is

$$\dot{x} = r\,x + h\,x^2 - x^3. \tag{3.2}$$

SHOW THIS by using a Taylor expansion in the regime where $(y,\,\lambda,\,\rho)$ are all small. ♣

Hint. The easiest approach is to expand f in powers of y, with coefficients that are functions of λ and ρ. Then use what you know of f to estimate the size of the coefficients (when λ and ρ are small), and then neglect any term that is majored by another term.† Then, upon re-scaling,‡ the resulting equation will be (3.2).

† A term neglected must be smaller than the terms retained for **all** $(y,\,\lambda,\,\rho)$ in some neighborhood of $(0,\,0,\,0)$. Making an expansion as suggested (as opposed to expanding in all three $(y,\,\lambda,\,\rho)$), simplifies this step quite a bit.

‡ Note that r and h are generally not directly λ and ρ, but functions of λ and ρ.

3.2 Answer: Perturbed pitchfork, with root preserved (canonical form)

First thing we need to do is compute a "leading order" Taylor approximation to f near $y = \lambda = \rho = 0$. We follow the hint and write

$$f = f(0,\,\lambda,\,\rho) + f_y(0,\,\lambda,\,\rho)\,y + \tfrac{1}{2}\,f_{yy}(0,\,\lambda,\,\rho)\,y^2 + \tfrac{1}{6}\,f_{yyy}(0,\,\lambda,\,\rho)\,y^3 + \dots \tag{3.3}$$

Now, using the properties of f we see that

1. *The first coefficient vanishes identically:* $f(0,\,\lambda,\,\rho) = 0$
2. *The second coefficient is small, of size* $O\left(\max(|\lambda|,\,|\rho|)\right)$.
   This follows because $f_y(0,\,0,\,0) = 0$, the condition for a bifurcation.
3. *The third coefficient is small, of size* $O\left(\max(|\lambda|,\,|\rho|)\right)$.
   This follows because $f_{yy}(0,\,0,\,0) = 0$, a consequence of $f(y,\,\lambda,\,0)$ being odd.
4. *The second and third coefficients are independent of each other, in general. Thus, through cancellations, one could be much smaller than the other. All we know is that their size is bounded by* $O\left(\max(|\lambda|,\,|\rho|)\right)$.
5. *The fourth coefficient is not small (stays away from zero), and it is negative.*
   This follows from the generic assumption that anything not specifically known to be zero, is non-zero. Thus $f_{yyy}(0,\,0,\,0) \neq 0$, so that $f_{yyy}(0,\,\lambda,\,\rho) \neq 0$ for λ and ρ small. In fact $f_{yyy}(0,\,0,\,0) < 0$, because the bifurcation is a soft pitchfork.
6. *From item 5 it follows that all the terms beyond* $O(y^3)$ *in* (3.3) are higher order.

Thus we approximate the equation by

$$\dot{y} = a_1\,y + a_2\,y^2 + a_3\,y^3, \tag{3.4}$$

where $a_1 = f_y(0,\,\lambda,\,\rho)$ (small), $a_2 = \frac{1}{2}\,f_{yy}(0,\,\lambda,\,\rho)$ (small), and $a_3 = \frac{1}{6}\,f_{yyy}(0,\,\lambda,\,\rho) < 0$. Now **define:**

$$\boldsymbol{x = \sqrt{|a_3|}\,y, \quad r = a_1 \quad \text{and} \quad h = a_2\,/\sqrt{|a_3|}.}$$

This transforms (3.4) into (3.2). ♣

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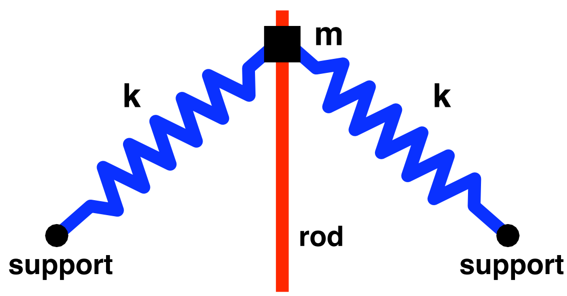
[4] That is, near the bifurcation, the full problem can be mapped into equation (3.2).

4 Toy model for shell buckling

4.1 Statement: Toy model for shell buckling

Hold a ping-pong ball between your thumb and index fingers and squeeze it. If you do not apply enough force, the ball will deform slightly with a purely elastic response. But, if you push hard enough, the ball will buckle and you will make a (permanent) dent on it — and the ball will be ruined. This is the phenomena of (thin) shell buckling.

Shell buckling is a very rich phenomena,[5] way beyond the scope of this course. Here we will study an extremely simplified (1-D) version of this phenomena (the emphasis here being on "toy" model) where all the geometrical richness of the original setting is gone, and only the buckling bifurcation remains.



A bead of mass m (black square) can slide along a rigid vertical rod (in red). The bead is connected by two equal springs (in blue), with spring constant k, to two supports placed symmetrically on each side of the rod. See the text for further details.

Figure 4.1: Toy model for shell buckling.

A sketch depicting the model is shown in figure 4.1. Further assumptions and notation are:

1. Idealize the bead as a point mass.
2. Let x be the vertical distance, along the rod, of the bead from the horizontal line joining the spring supports. Let $x > 0$ if the bead is above the supports and $x < 0$ if below.
3. Let $\boldsymbol{h > 0}$ be the distance of the spring supports from the rod, and let $\boldsymbol{L > 0}$ be the springs equilibrium length. **Assume $\boldsymbol{L > h}$**, so that the springs are under compression for $x = 0$.
4. Hook's law applies to the springs. Thus they exert a force of magnitude $F = k\,(\ell - L)$, where ℓ is the spring length, along the spring axis, pushing if $\ell < L$, and pulling if $\ell > L$.
5. When the bead slides along the rod, the motion is opposed by a friction force of magnitude $b\,\dot{x}$, where $b > 0$ is a constant.
6. Because the rod is rigid, we need to consider only the vertical components of the various forces that act on the bead. These forces are: (i) Gravity, of magnitude $m\,g$, pointing down. (ii) The forces by the springs. (iii) Friction along the rod. *Note that here we assume that the force gravity is significant, so that there is no up-down symmetry in this problem.*

PROBLEM TASKS:

A. *Derive an ode for the bead position, and write it in appropriate a-dimensional variables.*[6]

B. *Assume that friction is large, so that inertia can be neglected. Exactly which a-dimensional number has to be small for friction to be "large"?*

[5] Lots of interesting and important questions arise. For example: What is the shape of the dent that forms? The dent's edges have sharp corners: why these corners form, and how do they propagate as further pressure is applied?

[6] Suggestion: to a-dimensionalize use h for length and $b/(2\,k)$ for time.

C. *Analyze the bifurcations that occur for the equation resulting from item B, as the bead mass changes — in this toy model, increasing the bead mass plays the role of squeezing harder on the ping-pong ball. What type of bifurcation(s) occur?*

Hint: It is a bad idea to try to do this by attempting to solve for the critical points and bifurcation thresholds analytically. A qualitative, graphical, analysis is the best way to go.

D. *The picture in figure* 4.1 *corresponds, in this toy model, to the ping-pong ball in a more-or-less spherical shape. What is the "buckled" state?*

E. *What a-dimensional parameter controls when bifurcations happen?* This under the assumption:

$$\textbf{The ratio } \gamma = L/h > 1 \textbf{ is kept fixed.} \tag{4.1}$$

Thus γ is **not** the bifurcation parameter to use; something else is.

4.2 Answer: Toy model for shell buckling

Newton's law for the motion of the bead takes the form

$$m\,\ddot{x} + b\,\dot{x} = -m\,g + 2\,k\,\frac{x}{\sqrt{x^2+h^2}}\left(L - \sqrt{x^2+h^2}\right), \tag{4.2}$$

where the factor $2\,k$ arises because there are two springs, and the factor $x/\sqrt{x^2+h^2}$ is to compute the projection along the rod of the spring's forces. Note also that the signs are correct: when the springs are under compression ($\sqrt{x^2+h^2} < L$), and $x > 0$, the springs should be pushing x up — with the force sign switching if either $x < 0$ or $\sqrt{x^2+h^2} > L$.

Select a-dimensional variables via $\boldsymbol{x = h\,\tilde{x}}$ and $\boldsymbol{t = \dfrac{b}{2\,k}\,\tilde{t}}$. The equation then becomes

$$\epsilon\,\ddot{x} + \dot{x} = -r + \frac{x}{\sqrt{1+x^2}}\left(\gamma - \sqrt{1+x^2}\right), \tag{4.3}$$

where we have not written the tildes to simplify the notation,

$$\epsilon = \frac{2\,k\,m}{b^2}, \quad \text{and} \quad r = \frac{m\,g}{2\,k\,h}. \tag{4.4}$$

If $\epsilon \ll 1$, we can neglect inertia. Thus we arrive at the final equation (the toy model equation)

$$\dot{x} = -r + \underbrace{\frac{x}{\sqrt{1+x^2}}\left(\gamma - \sqrt{1+x^2}\right)}_{p=p(x)} = p(x) - r. \tag{4.5}$$

Since γ is kept fixed, **the bifurcation parameter is r.** To understand the critical point structure of this equation, in figure 4.2 we plot $y = p(x)$ and $y = r$ — this for some value of γ (there is no qualitative difference if γ is changed). **Let $r_c > 0$ be the value of p at the (single) local maximum for $x > 0$** — that is $\boldsymbol{r_c = p(x_c)}$, **where x_c is the location at which the local maximum occurs.** Note that here we operate as if γ were a fixed constant, but (in fact) both r_c and x_c are functions of γ — which must be computed numerically, if needed.

Three cases arise:

c1. Case $0 < r < r_c$. Three critical points: x_1, x_2, and x_3 — which satisfy $x_1 < 0 < x_2 < x_c < x_3 < \sqrt{\gamma^2 - 1}$. Both x_1 and x_3 are stable, while x_2 is unstable.
- x_3 *corresponds to the configuration in figure* 4.1, with the bead being supported by the two (compressed) springs above the level $x = 0$.
- x_1 *corresponds to a configuration where the bead is hanging from the two (stretched) springs.* As follows from item c3, **this is the "buckled" state in this model.**

c2. Case $r_c < r$. Only one critical exists: the "buckled" state x_1 — which is stable. [7]

[7] Because we are assume a situation where gravity matters, there is no truly "un-buckled" state — at best a slightly deformed one.

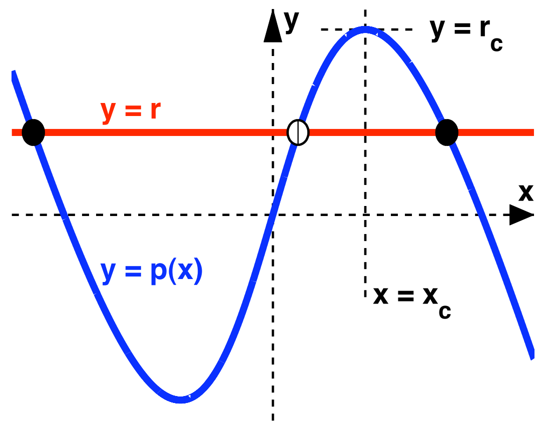


Figure 4.2: Critical points for the toy model for shell buckling. The critical points occur at the values of x where $y = r$ intersects $y = p(x)$ — where p is defined in equation (4.5).

c3. Case $r_c = r$. Critical threshold value at which a saddle-node bifurcation occurs. As r increases through r_c, x_3 looses stability, and the system jumps to x_1 (if it was in x_3).

5 Problem 03.02.06 - Strogatz (Eliminate the cubic term)

5.1 Statement for problem 03.02.06

Consider the system

$$\frac{dX}{dt} = RX - X^2 + aX^3 + O(X^4), \tag{5.6}$$

where $R \neq 0$. We want to find a new variable x such that the system transforms into

$$\frac{dx}{dt} = Rx - x^2 + O(x^4). \tag{5.7}$$

This would be a big improvement, since the cubic term has been eliminated and the error term has been bumped to fourth order.[8] In fact, the procedure to do this (sketched below) can be generalized to higher orders.[9] This generalization is the subject matter of problem 03.02.07.

Let $\boldsymbol{x = X + bX^3 + O(X^4)}$, where b is chosen later to eliminate the cubic term in the differential equation for x. This is called a **near-identity transformation**, since x and X are practically equal: they differ by a cubic term.[10] Now we need to rewrite the system in terms of x; this calculation requires a few steps.

1. Show that the near-identity transformation can be inverted to yield $\boldsymbol{X = x + cx^3 + O(x^4)}$, and solve for c.

2. Write $\boldsymbol{\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)}$, and substitute for X and $\dot{X}$ on the right hand side, so that everything depends only on x. Multiply the resulting series expansions and collect terms, to obtain $\boldsymbol{\dot{x} = Rx - x^2 + kx^3 + O(x^4)}$, where k depends on a, b, and R.

[8] Obviously we are considering here a situation where X (and x) is small.
[9] That is, one can successively eliminate all the higher order terms: $O(x^3)$, $O(x^4)$, ..., etc.
[10] We have skipped the quadratic term X^2, because it is not needed — you should check this later.

3. Now the moment of triumph: choose b so that $k = 0$.
4. Is it really necessary to make the assumption that $R \neq 0$? Explain.

5.2 Answer for problem 03.02.06

We now fill in the steps outlined in the problem statement:

1. Replacing $X = x + cx^3 + O(x^4)$ into $x = X + bX^3 + O(X^4)$ yields:

   $x = (x + cx^3) + b(x + cx^3)^3 + O(x^4) = x + (c + b)x^3 + O(x^4)$. Thus, it must be $\boldsymbol{c = -b}$.

   This process can be carried out to any order. If $x = X + aX^2 + bX^3 + cX^4 + \cdots + O(X^N)$, we can find the inverse transformation $X = x + Ax^2 + Bx^3 + Cx^4 + \cdots + O(x^N)$ by successively selecting the coefficients A, B, C, ... to eliminate the coefficients of the powers x^2, x^3, x^4, ... in a substitution like the one above.

2. Write $\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)$, use equation (5.6) to eliminate $\dot{X}$ on the right hand side, and substitute $X = x - bx^3 + O(x^4)$ — as obtained in the first step — to eliminate X. This yields:

$$\begin{aligned}
\dot{x} &= \dot{X} + 3bX^2\dot{X} + O(X^4) \\
&= (RX - X^2 + aX^3) + 3bX^2(RX - X^2 + aX^3) + O(X^4) \\
&= RX - X^2 + (a + 3bR)X^3 + O(X^4) \\
&= R(x - bx^3) - (x - bx^3)^2 + (a + 3bR)(x - bx^3)^3 + O(x^4) \\
&= Rx - x^2 + kx^3 + O(x^4), \quad \text{where } \boldsymbol{k = a + 2\,b\,R}.
\end{aligned}$$

3. Now choose b so that $k = 0$. That is
$$\boldsymbol{b = -\frac{a}{2\,R}}. \tag{5.8}$$

4. Equation (5.8) shows that $R \neq 0$ is crucial for all of this to work. When $R = 0$, X^2 is the *dominant* term on the right in (5.6), and the proposed form of the expansion does not work. It is **still possible to eliminate the $\boldsymbol{O(X^3)}$ term in (5.6) — as well as any other higher order terms — when $\boldsymbol{R = 0}$, but a DIFFERENT EXPANSION IS NEEDED, including logarithmic terms.** The first two terms in this expansion are: $\boldsymbol{x = X + a\,X^2\,\ln X + \ldots}$

Remark 5.5 *What would have happened if we started with a more general form of the transformation relating x and X, that is: $x = X + qX^2 + bX^3 + O(X^4)$? Then, in the second step above the final answer would have taken the form $\dot{x} = Rx - px^2 + kx^3 + O(x^4)$. Then the next step would have been to select q and b so that $p = 1$ and $k = 0$. This would have given $q = 0$ and $k = -a/2R$. That is: the same answer as above. We have simplified the algebra by taking $q = 0$ from the very beginning.*

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6 Stability index for flows in the circle

6.1 Statement: Stability index for flows in the circle

Show that *the stability index $\mathcal{S}$ for any flow in the circle vanishes.* To be precise, consider an equation of the form

$$\frac{d\theta}{dt} = f(\theta), \tag{6.1}$$

where θ is an angle (in radians), and f is periodic of period $2\,\pi$ and Lipschitz continuous. Assume also that the equation has a finite number of critical points:[11] $\theta_1 < \theta_2 < \cdots < \theta_N < \theta_1 + 2\,\pi$. Now assign a weight $w = 1$ to each

[11] The critical points are the zeros of f.

stable critical point, a weight $w = -1$ to each unstable critical point, and a weight $w = 0$ to each semi-stable critical point. Then show that

$$\mathcal{S} = \sum_{n=1}^{N} w_n = 0. \tag{6.2}$$

Hint 6.1 *Consider the intervals I_n, $1 \le n \le N$, where I_n is the interval $\theta_n < \theta < \theta_{n+1}$ — here $\theta_{N+1} = \theta_1 + 2\pi$, which is the same point as θ_1 because we are in the circle. Then in each such interval either*[12] *$f > 0$ or $f < 0$.* **Define** *$\sigma_n = 1$ if $f > 0$ in I_n, and $\sigma_n = -1$ if $f < 0$ in I_n. Then relate the w_n to the σ_n to show (6.2).* What information do the σ_n capture? ♣

6.2 Answer: Stability index for flows in the circle

The σ_n characterize the direction of the flow given by (6.1). If $\sigma_n = 1$, then the flow is from θ_n towards θ_{n+1}. If $\sigma_n = -1$, the flow is in the opposite direction. It is then easy to see that

$$w_n = \frac{1}{2}(\sigma_{n-1} - \sigma_n) \quad \text{for } \ 1 \le n \le N, \tag{6.3}$$

where $\sigma_0 = \sigma_N$ (again, we are in a circle, so that I_0 is the same as I_N). It follows that

$$\mathcal{S} = \frac{1}{2}\sum_{n=1}^{N}(\sigma_n - \sigma_{n-1}) = \frac{1}{2}\sum_{n=1}^{N}\sigma_n - \frac{1}{2}\sum_{0}^{N-1}\sigma_n = 0, \tag{6.4}$$

where we have used that $\sigma_0 = \sigma_N$.

7 Bifurcations in the circle problem #06

7.1 Statement: Bifurcations in the circle problem #06

For equation (7.1) find the values of r at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram for the fixed points versus r, including the flow direction and the stability of the various branches of solutions (solid lines for stable branches and dashed ones for unstable ones).

$$\frac{d\theta}{dt} = (r + \sin(2\theta))\,\sin(\theta), \tag{7.1}$$

where θ is an angle (in radians). Note that *the bifurcation diagram — which is periodic in θ — should be for a 2π range in θ, and a range of r that includes all the bifurcations.*

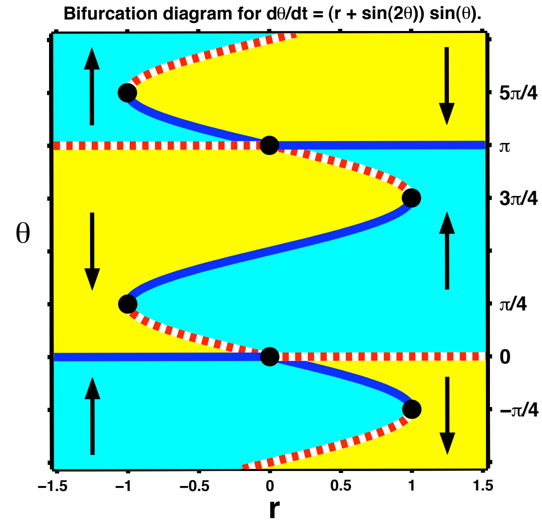
7.2 Answer: Bifurcations in the circle problem #06

The critical points for equation (7.1) are given by $\theta = n\pi$ (n an integer), and the solutions to the equation

$$r = -\sin(2\theta). \tag{7.2}$$

Equation (7.2) has no solutions for $r < -1$ or $r > 1$, and four solutions (per 2π-period) for $-1 < r < 1$. The bifurcations occur for the values of r at which the number of solutions changes: $r = \pm 1$, as well as $r = 0$ — where

[12] If f were to switch sign in I_n, then (since it is continuous) it would have a zero in I_n. This zero would no be one of the θ_n, which are supposed to be all the zeros.



In each region (yellow or cyan), the black arrows indicate the direction of the flow for the equation $\dot{\theta} = (r + \sin(2\,\theta))\sin(\theta)$.

Stable branches of critical points are plotted in solid blue, and unstable branches in dashed red. The black dots indicate the bifurcation points:
- Saddle node at $r = -1$ and $\theta = \frac{1}{4}\pi + n\,\pi$.
- Saddle node at $r = +1$ and $\theta = \frac{3}{4}\pi + n\,\pi$.
- Transcritical at $r = 0$ and $\theta = n\,\pi$.

Figure 7.1: Bifurcation diagram for equation 7.1.

there are crossings of critical point curves. By looking at the sign of $\dot{\theta}$ in the regions into which the r-θ plane is divided by the curves $r = -\sin(2\,\theta)$ and $\theta = n\,\pi$, it is easy to ascertain the stability of the critical points, as well as the nature of the various bifurcations that occur. The results are summarized in figure 7.1.

– **Saddle node** bifurcations occur at $\boldsymbol{r = -1}$ and $\boldsymbol{\theta = \frac{1}{4}\pi + n\,\pi}$, n integer.

– **Saddle node** bifurcations occur at $\boldsymbol{r = 1}$ and $\boldsymbol{\theta = \frac{3}{4}\pi + n\,\pi}$, n integer.

– **Transcritical** bifurcations occur at $\boldsymbol{r = 0}$ and $\boldsymbol{\theta = n\,\pi}$, n integer.

8 Bifurcations in the torus #01

8.1 Statement: Bifurcations in the torus #01

Bifurcations in the torus, phase-locking, and oscillator death. This problem is based on a paper on systems of neural oscillators by G. B. Ermentrout and N. Kopell, [13] where they illustrate the notion of *oscillator death* (see § 10) with the following model

$$\dot{\theta}_1 = \omega_1 + \sin\theta_1\,\cos\theta_2 \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + \sin\theta_2\,\cos\theta_1, \tag{8.1}$$

where $\boldsymbol{\omega_1,\, \omega_2 > 0}$. *Here θ_1 and θ_2 are to be interpreted as the phases of two coupled stable and attracting limit cycle oscillators, which are assumed to "survive" the coupling, so that the notion of their "individual phases" remains —* see § 10.

a. Classify all the different behaviors that the solutions to (8.1) have, as the parameters vary in the positive quadrant of the $[\omega_1,\, \omega_2]$–plane. **Do a diagram** in this quadrant, indicating the regions that correspond to each behavior.

The final answer should look something like this: *(i) In such and such region the solutions are attracted to a limit cycle [Note that this is phase locking]. (ii) In such and such region the solutions are attracted to a stable*

[13] *Oscillator death in systems of coupled neural oscillators.* SIAM J. Appl. Math. **50**:125 (1990).

node [Note that this is oscillator death]. (iii) In such and such region the solutions are quasi-periodic with two periods [Phase locking fails]. (iv) ...
Plus a drawing of the regions ... will all the statements properly justified.

b. Draw the bifurcation curves in the $[\omega_1,\, \omega_2]$–plane. **Describe** each bifurcation.

Hints. *I did not find an elegant way to analyze the system geometrically. The hints below lead you to an approach that is (mostly) analytical, but allows a systematic and thorough investigation.*

h1. Consider the equations satisfied by $\phi = \theta_1 + \theta_2$ and $\psi = \theta_1 - \theta_2$.

h2. You may find the following result useful
Let $\boldsymbol{\alpha > 1}$. Then the solutions to the equation $\dot{\boldsymbol{\chi}} = \boldsymbol{\alpha + \sin\chi}$
can be written in the form $\boldsymbol{\chi = \mu\,(t - t_0) + X(\mu\,(t - t_0))}$,
where $\boldsymbol{\mu > 0}$ is a constant, $\boldsymbol{X}$ is $2\,\pi$-periodic,
$\boldsymbol{X(0) = 0}$, and t_0 is an arbitrary constant.

Furthermore: $\boldsymbol{\mu}$ is an increasing function of $\boldsymbol{\alpha}$, with $\displaystyle\lim_{\alpha\to 1} \boldsymbol{\mu = 0}$ and $\displaystyle\lim_{\alpha\to\infty} \boldsymbol{\mu = \infty}$.

All this follows from § 11, upon using a change of variables to transform $\dot{\chi} = \alpha + \sin\chi$ into equation (11.1). In particular, note that the "μ" in § 11 (call it $\tilde{\mu}$) is related to the one here by $\mu = \alpha\,\tilde{\mu}$, with $\kappa = \frac{1}{\alpha}$.

8.2 Answer: Bifurcations in the torus #01. Answer

With $\boldsymbol{\phi = \theta_1 + \theta_2}$ and $\boldsymbol{\psi = \theta_1 - \theta_2}$, standard trigonometric formulas show that the system has the equivalent form

$$\dot{\boldsymbol{\phi}} = \boldsymbol{\omega_1 + \omega_2 + \sin\phi} \quad \text{and} \quad \dot{\boldsymbol{\psi}} = \boldsymbol{\omega_1 - \omega_2 + \sin\psi}. \tag{8.2}$$

Hence the system is **uncoupled** in these variables. It follows that (see figure 8.1)

1. **Case $\boldsymbol{\omega_1 > \omega_2 + 1}$. Oscillators "independent", The solutions are quasi-periodic, with two periods.** Since $\omega_1 + \omega_2 > \omega_1 - \omega_2 > 1$, from item **h2** in the hints [14]

$$\phi = \mu_1\, t + \Phi \quad \text{and} \quad \psi = \mu_2\, t + \Psi, \quad \text{where } \mu_1 > \mu_2 > 0, \tag{8.3}$$

Φ is periodic of period $\boldsymbol{T_1 = \frac{2\,\pi}{\mu_1}}$, and Ψ is periodic of period $\boldsymbol{T_2 = \frac{2\,\pi}{\mu_2}}$.

There is **neither phase locking, nor oscillator death.** Each oscillator oscillates independently, albeit with greatly modified phases:

$$\theta_1 = \frac{\mu_1 + \mu_2}{2}\, t + \frac{1}{2}\,(\Phi + \Psi) \quad \text{and} \quad \theta_2 = \frac{\mu_1 - \mu_2}{2}\, t + \frac{1}{2}\,(\Phi - \Psi), \tag{8.4}$$

instead of the uncoupled phases $\theta_j = \omega_j\,(t - t_{0\,j})$. The frequencies (see § 10) are not constant either, that is: $\dot{\boldsymbol{\theta}}_{\boldsymbol{1}} = \boldsymbol{\frac{\mu_1+\mu_2}{2} + \frac{1}{2}\,(\dot{\Phi} + \dot{\Psi})}$ and $\dot{\boldsymbol{\theta}}_{\boldsymbol{2}} = \boldsymbol{\frac{\mu_1-\mu_2}{2} + \frac{1}{2}\,(\dot{\Phi} - \dot{\Psi})}$.

Finally, note that *the solutions are quasi-periodic, with periods T_1 and T_2.*
When checking periodicity, or quasi-periodicity, it is important to keep in mind that θ_1 and θ_2 are angles, so that behavior of the form $\theta_j = \mu\, t$ corresponds to a period $T = 2\,\pi/\mu$.

2. **Case $\boldsymbol{\omega_1 = \omega_2 + 1}$. Bifurcation into (out of) phase locking.** Φ has the same form as in (8.3), but the equation for Ψ in (8.2) has a semi-stable critical point $\Psi = \pi/2 + 2\,n\,\pi$. *Phase locking is semi-stable:* small perturbations can take the system out of phase lock. Then the phase Ψ difference increases by $2\,\pi$, and phase locking occurs again.

[14] To be precise: $\Phi(t) = X_1(\mu_1(t - t_{01})) - \mu_1\, t_{01}$ and $\Psi(t) = X_2(\mu_2(t - t_{02})) - \mu_2\, t_{02}$, in the hints' notation.

Consider the situation in item **1** *as* $\omega_1 - \omega_2 \downarrow 1$. Then $\mu_2 \downarrow 0$ and $T_2 \uparrow \infty$. Hence ψ becomes very slowly varying — a constant for "short enough" time periods, and the oscillator's behavior approximates that of a phase locked one, with a a common frequency.

Consider the situation in item **3** *as* $\omega_1 - \omega_2 \uparrow 1$. Then ψ_u and ψ_s coalesce into $\psi_u = \psi_s = \pi/2$, and the behavior limits to the one here.

3. **Case $|-\omega_2 + 1| < \omega_1 < \omega_2 + 1$. Phase lock is a global attractor.** Since $|\omega_1 - \omega_2| < 1$, the equation for ψ in (8.2) has two critical points, one unstable (call it ψ_u), and another a stable global attractor (call it ψ_s). Thus a constant phase difference $\psi = \psi_s$ is a global attractor. On the other hand $\omega_1 + \omega_2 > 1$, and item **h2** in the hints yields
$$\phi = \mu_1\, t + \Phi \quad \text{where } \mu_1 > 0 \text{ and } \Phi \text{ is periodic of period } T_1 = \frac{2\,\pi}{\mu_1}. \tag{8.5}$$
Thus, the attracting solution is
$$\theta_j = \frac{1}{2}\,\mu_1\, t + \frac{1}{2}\,(\Phi - (-1)^j\,\psi_s), \tag{8.6}$$
so that the two oscillators run with a common (variable) frequency $\boldsymbol{\omega = \frac{1}{2}\,\mu_1 + \frac{1}{2}\,\dot{\Phi}}$. *This solution is limit cycle, with period* $\boldsymbol{T_1}$

4. **Case $\omega_1 + \omega_2 = 1$. Bifurcation from/to oscillator death to/from phase locking.** The equation for ψ in (8.2) has "the same" two critical points, ψ_u and ψ_s, in item **3**, while the equation for ϕ has a the single, semi-stable, critical point $\pi/2 + 2\,n\,\pi$. The system then approaches, as $t \to \infty$, the critical point $\phi = \pi/2 + 2\,n\,\pi$ and $\psi = \psi_s$. But this point is semi-stable, so a small perturbation can take ϕ out of equilibrium. Then ϕ increases by $2\,\pi$, till it reaches the critical point again ($n \to n+1$).

   The case in item **5** *approaches the behavior here as* $\omega_1 + \omega_2 \downarrow 1$,
   because then $\mu_1 \downarrow 0$ and $T_1 \uparrow \infty$.
   Hence ϕ becomes very slowly varying — a constant for "short enough" time periods.

   The case in item **5** *approaches the behavior here as* $\omega_1 + \omega_2 \uparrow 1$,
   because then ϕ_u and ϕ_s coalesce into $\pi/2 + 2\,n\,\pi$.

5. **Case $\omega_1 + \omega_2 < 1$. Oscillator death.** Since the ω_j are positive, $|\omega_1 - \omega_2| < 1$. Thus both equations in (8.2) have critical points, one unstable in each (ϕ_u & ψ_u), and another a stable global attractor in each (ϕ_s & ψ_s). Thus *the system in (8.1) has a global attractor (a stable node), given by*
$$(\theta_1,\, \theta_2) = \frac{1}{2}\,(\phi_s + \psi_s,\, \phi_s - \psi_s). \tag{8.7}$$
Note: $\frac{1}{2}\,(\phi_s + \psi_u,\, \phi_s - \psi_u)$ & $\frac{1}{2}\,(\phi_u + \psi_s,\, \phi_u - \psi_s)$ are saddles, while $\frac{1}{2}\,(\phi_u + \psi_u,\, \phi_u - \psi_u)$ is an unstable node.

6. **Case $\omega_2 = \omega_1 + 1$. Bifurcation into (out of) phase locking.** This case is the same as the one in item **2**, via the system symmetry $\theta_1 \leftrightarrow \theta_2$.

7. **Case $\omega_2 > \omega_1 + 1$. Oscillators "independent", The solutions are quasi-periodic, with two periods.** This case is the same as the one in item **2**, via the system symmetry $\theta_1 \leftrightarrow \theta_2$.

To finish with the problem answer, next we prove the statements in item **h2** of the hints, using the results from § 11.

Lemma 8.1 *Let $\alpha > 1$. Then the solutions to the equation*
$$\dot{\chi} = \alpha + \sin\chi, \tag{8.8}$$
have the form
$$\chi = \mu\,(t - t_0) + X(\mu(t - t_0)), \tag{8.9}$$
where $0 < \mu < \alpha$ is a constant, X is periodic of period $2\,\pi$, $X(0) = 0$, and t_0 is some constant. Furthermore: μ is an increasing function of α such that:

(i) For $0 < \alpha - 1 \ll 1$, $\mu = O\big(\sqrt{\alpha - 1}\big)$. *(ii) For $\alpha \gg 1$, $\mu/\alpha \sim 1$.*

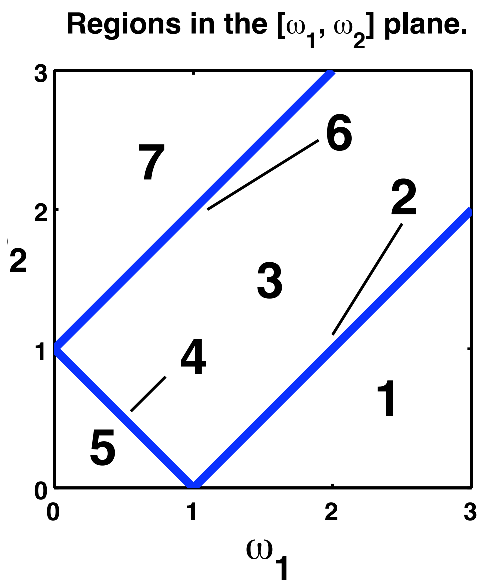


Figure 8.1: (Problem 08.06.01). *Oscillator death and bifurcations in the plane.* Regions in the positive quadrant of the $[\omega_1,\, \omega_2]$–plane corresponding to different behaviors (8.1) can have. The boundaries between the regions correspond to bifurcations. These are:

2, where $\boldsymbol{\omega_1 - \omega_2 = 1}$;
4, where $\boldsymbol{\omega_1 + \omega_2 = 1}$; and
6, where $\boldsymbol{\omega_2 - \omega_1 = 1}$.

Consider equation (11.1) with $\boldsymbol{\kappa = 1/\alpha}$*. Then we can write*

$$\phi(\hat{t}) = \hat{\mu}\,(\hat{t} - \hat{t}_0) + \Phi\left(\hat{\mu}\,(\hat{t} - \hat{t}_0)\right), \tag{8.10}$$

where $\hat{\mu}$ *and* $\hat{\Phi}$ *have the properties stated in § 11. Now let*

$$\hat{t} = -\alpha\, t, \quad \hat{t}_0 = -\alpha\, t_0, \quad \mu = \alpha\,\hat{\mu}, \quad \chi(t) = -\phi(\,\hat{t}\,) \quad \textit{and} \quad X(\zeta) = -\Phi(-\zeta). \tag{8.11}$$

Then (8.10) reduces to (8.9). Furthermore, via the transformation $\chi(t) = -\phi(-\alpha\, t)$*, (11.1) and (8.8) are equivalent. Then all the properties in the statement of the lemma follow from § 11, except for one: That* μ *is an increasing function of* α*.*
To show that μ is an increasing function of α, *we proceed as follows. Let* χ_* *be the solution obtained by setting* $t_0 = 0$ *in (8.9). Then* $\chi_*(T) = 2\,\pi$*, where* $T = 2\,\pi/\mu$*. Furthermore, since* $\chi_*(0) = 0$ *and* χ_* *is strictly increasing, this defines* T *uniquely. Then, from (8.8), it should be obvious that* T *is a strictly decreasing function of* α*.* **Q.E.D.**

9 First order equation with a periodic right hand side

9.1 Statement: First order equation with a periodic right hand side

Consider the equation

$$\boldsymbol{\dot{\phi} = 1 - \kappa\, \sin\phi,} \quad \text{where} \quad \boldsymbol{0 < \kappa < 1.} \tag{9.1}$$

Since $\dot{\phi} \geq 1 - \kappa > 0$, ϕ *is monotone increasing.* **Prove the statements below.**

1. There is a constant $0 < \mu < 1$, and a function $\Phi = \Phi(\zeta)$ — periodic of period $2\,\pi$ — such that any solution to (9.1) has the form
$$\boldsymbol{\phi = \mu\,(t - t_0) + \Phi(\mu\,(t - t_0)),} \tag{9.2}$$
where t_0 is a constant and $\boldsymbol{\Phi(0) = 0}$. It follows that $\boldsymbol{\sin(\phi)}$ **is periodic in** $\boldsymbol{t}$**, with period** $\boldsymbol{T = \frac{2\,\pi}{\mu}}$
2. The period-average M for $\sin(\phi)$ is given by $\quad \boldsymbol{M =}$ **average**$\boldsymbol{(\sin\phi) = \frac{1-\mu}{\kappa} > 0}$. (9.3)
$M = M(\kappa)$ only (μ depends only on κ).

3. Let $\boldsymbol{\phi_*}$ **be the solution to (9.1) defined by** $\boldsymbol{\phi_*(0) = 0}$ — i.e.: set $t_0 = 0$ in (9.2). Then

$$\Theta(\mu\, t) = \int_0^t (\sin(\phi_*(s)) - M)\, ds = -\frac{1}{\kappa}\, \Phi(\mu\, t), \tag{9.4}$$

where Θ is defined by the first equality.

4. Assume that $0 < \kappa \ll 1$. Then a Poincaré-Lindstedt expansion yields

$$\phi_* = \mu\, t - \kappa\,(1 - \cos(\mu\, t)) + O(\kappa^2) \quad \text{and} \quad \mu = 1 - \frac{1}{2}\,\kappa^2 + O(\kappa^4). \tag{9.5}$$

It follows that $T = 2\,\pi + \pi\,\kappa^2 + O(\kappa^4)$ and $M = \dfrac{1}{2}\,\kappa + O(\kappa^3)$.

5. Assume that $0 < 1 - \kappa \ll 1$. Then

$$\boldsymbol{\mu = O(\sqrt{1-\kappa})} \tag{9.6}$$

Hints.

a. Define $T > 0$ as the (unique) time at which $\phi_*(T) = 2\,\pi$ — why is the solution unique?

b. Show that $\phi_*(t+T) = 2\,\pi + \phi_*(t)$ — sub-hint: both sides are solutions!

c. Define Φ by $\Phi(\mu\, t) = \phi_*(t) - \mu\, t$, with $\mu = 2\,\pi/T$, and show that Φ is periodic of period $2\,\pi$.

d. Write the general solution in terms of ϕ_*.

e. Show that $T = O\big(1/\sqrt{1-\kappa}\big)$ as $\kappa \to 1$ — sub-hint: critical slowing-down.

f. To show that $\mu < 1$, use (9.1) and separation of variables to write T as an integral over ϕ from 0 to $2\,\pi$. Then show $T > 2\,\pi$

g. To show (9.3), take the average of (9.1).

h. To obtain the second equality in (9.4), substitute $\phi_* = \mu\, t + \Phi(\mu\, t)$ into (9.1), and obtain a formula for $\sin(\phi_*)$ in terms of Φ.

9.2 Answer: First order equation with a periodic right hand side.

A. Since $\phi_*(0) = 0$ and $\dot{\phi}_* \geq 1 - \kappa > 0$, **there is a unique** $\boldsymbol{T > 0}$ at which $\boldsymbol{\phi_*(T) = 2\,\pi}$.

B. $\phi_1(t) = \phi_*(t+T)$ and $\phi_2(t) = 2\,\pi + \phi_*(t)$ are both solutions to (9.1), with the same initial data $\phi_1(0) = \phi_2(0) = 2\,\pi$. Thus $\phi_1 = \phi_2$, i.e.: $\boldsymbol{\phi_*(t+T) = 2\,\pi + \phi_*(t)}$ **for all** $\boldsymbol{t}$.

C. Define $\Phi = \Phi(\zeta)$ by $\boldsymbol{\Phi(\mu\, t) = \phi_*(t) - \mu\, t}$, with $\boldsymbol{\mu = 2\,\pi/T}$. Then: $\boldsymbol{\Phi}$ **has period** $\boldsymbol{2\,\pi}$.
Proof. $\Phi(\mu\, t + 2\,\pi) = \Phi(\mu\,(t+T)) = \phi_*(t+T) - \mu\,(t+T) = 2\,\pi + \phi_*(t) - \mu\, t - 2\,\pi = \Phi(\mu\, t)$, *where we have used the result in item* **B.** Note also that $\boldsymbol{\Phi(0) = 0}$.

D. Let $\boldsymbol{\phi}$ **be an arbitrary solution to (9.1).** Since $\dot{\phi} \geq 1 - \kappa > 0$, ϕ is strictly increasing and takes all values $\Rightarrow$ **there is a** $\boldsymbol{t_0}$ **such that** $\boldsymbol{\phi(t_0) = 0}$. Hence $\underline{\phi(t) = \phi_*(t - t_0)}$, since both sides are solutions to (9.1) with the same value at $t = t_0$. On the other hand, by the definition of Φ in item **C**, $\underline{\phi_*(t) = \mu\, t + \Phi(\mu\, t)}$. Thus, **equation (9.2) follows.**

E. Since $T > 0$, $\mu > 0$. Below we show $T > 2\,\pi$, thus completing the proof of $\boldsymbol{0 < \mu < 1}$.
Proof. Separate variables in (9.1), use that $\phi_*(0) = 0$, $\phi_*(T) = 2\,\pi$, *and integrate, to obtain*

$$T = \int_0^{2\,\pi} \frac{d\phi}{1 - \kappa\, \sin\phi} > \int_0^{2\,\pi} (1 + \kappa\, \sin\phi)\, d\phi = 2\,\pi.$$

Here the inequality in the middle follows because $(1-s)^{-1} < 1+s$ *for* $-1 < s < 1$. *Furthermore, integrate (9.1), and use the definition of* M, *to obtain*

$$2\pi = \int_0^T (1-\kappa\sin\phi)\,dt = T\,(1-\kappa\,M) \qquad \Longrightarrow \quad M = \frac{1-\mu}{\kappa}.$$

This proves (9.3).

F. Suppose that $0 < 1-\kappa \ll 1$. Then, when $\phi \approx \pi/2$, equation (9.1) has a *critical slow-down*, which takes a $\Delta t = O\big(1/\sqrt{1-\kappa}\big)$ to traverse. In addition, there are no other critical slow-downs for $0 \le \phi \le 2\pi$. Therefore $T = O\big(1/\sqrt{1-\kappa}\big)$, from which **(9.6) follows.**

G. Now we prove the second equality in (9.4).
Proof. Substitute $\phi_* = \mu\,t + \Phi(\mu\,t)$ *into the left hand side of (9.1), and solve for* $\sin\phi_*$. *This yields, after using the expression for* M *in (9.3),* $\sin(\phi_*) = M - \dfrac{1}{\kappa}\dfrac{d}{dt}\Phi(\mu\,t)$. *Substitute into the integral in (9.4), and use* $\Phi(0) = 0$, *to obtain the desired equality.*

H. Assume that $0 < \kappa \ll 1$, and expand $\phi*$ as follows (Poincaré-Lindstedt)

$$\phi_* = \mu\,t + \kappa\,\Phi_0(\mu\,t) + \kappa^2\,\Phi_1(\mu\,t) + \dots \quad \text{where} \quad \mu = 1 + \kappa\,\mu_1 + \kappa^2\,\mu_2 + \dots \tag{9.7}$$

and $\Phi_j(\zeta)$ is periodic of period 2π. Substitute this into (9.1), and solve order by order — determining μ_j by requiring Φ_j to be periodic. **This yields (9.5).**

Part I

Supplementary notes on oscillators and ode

10 Notes: coupled oscillators, phase locking, oscillator death, etc.

10.1 On phases and frequencies

Consider a system made by two coupled oscillators, where each of the oscillators (when not coupled) has a stable attracting limit cycle. Let the limit cycle solutions for the two oscillators be given by $\boldsymbol{\vec{x}_1 = \vec{F}_1(\omega_1\,t)}$ and $\boldsymbol{\vec{x}_2 = \vec{F}_2(\omega_2\,t)}$, where $\boldsymbol{\vec{x}_1}$ and $\boldsymbol{\vec{x}_2}$ are the vectors of variables for each of the two systems, the $\boldsymbol{\vec{F}_j}$ *are periodic functions of period* 2π, and the ω_j are constants (related to the limit cycle periods by $\omega_j = 2\pi/T_j$). In the un-coupled system, the two limit cycle orbits make up a *stable attracting invariant torus* for the evolution. Assume now that either the coupling is weak, or that the two limit cycles are strongly stable. *Then the stable attracting invariant torus survives for the coupled system.*[15] The solutions (on this torus) can be (approximately) represented by

$$\vec{x}_1 \approx \vec{F}_1(\theta_1) \quad \text{and} \quad \vec{x}_2 \approx \vec{F}_2(\theta_2), \tag{10.1}$$

where $\boldsymbol{\theta_1 = \theta_1(t)}$ and $\boldsymbol{\theta_2 = \theta_2(t)}$ satisfy some equations, of the general form

$$\dot{\theta}_1 = \omega_1 + K_1(\theta_1,\,\theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + K_2(\theta_1,\,\theta_2). \tag{10.2}$$

Here K_1 and K_2 are the "projections" of the coupling terms along the oscillator limit cycles. For example, take $K_1(\theta_1,\,\theta_2) = \sin\theta_1\,\cos\theta_2$ and $K_2(\theta_1,\,\theta_2) = \sin\theta_2\,\cos\theta_1$. Another example is the one in § 8.6 of Strogatz' book (*Nonlinear Dynamics and Chaos*), where a model system with

[15] With a (slightly) changed shape and position.

$$K_1(\theta_1,\,\theta_2) = -\kappa_1\,\sin(\theta_1-\theta_2) \text{ and } K_2(\theta_1,\,\theta_2) = \kappa_2\,\sin(\theta_1-\theta_2)$$

is introduced, with constants $\kappa_1,\,\kappa_2 > 0$. Note that:

1. In (10.2), K_1 and K_2 must be $2\,\pi$-periodic functions of θ_1 and θ_2.
2. The **phase space for (10.2) is the invariant torus $\mathcal{T}$**, on which θ_1 and θ_2 are the angles. We can also think of $\mathcal{T}$ as a $2\,\pi \times 2\,\pi$ square with its opposite sides identified. On $\mathcal{T}$ a **solution is periodic** if and only if $\theta_1(t+T) = \theta_1(t) + 2\,n\,\pi$ and $\theta_2(t+T) = \theta_2(t) + 2\,m\,\pi$, where $T > 0$ is the period, and both n and m are integers.
3. In the "Coupled oscillators # 01" problem an example of the process leading to (10.2) is presented.
4. The $\boldsymbol{\theta_j}$'s are the **oscillator phases.** One can also define oscillator frequencies, even when the θ_j's do not have the form $\theta_j = \omega_j\,t$, with ω_j constant.
   The idea is that, near any time t_0 we can write $\theta_j = \theta_j(t_0) + \dot{\theta}_j(t_0)\,(t-t_0) + \ldots$, identifying $\dot{\theta}_j(t_0)$ as the local frequency. Hence, we define the **oscillator frequencies by** $\boldsymbol{\tilde{\omega}_j = \dot{\theta}_j}$. These frequencies are, of course, **generally not constants.**
5. The notion of phases can survive **even if** the limit cycles cease to exist (i.e.: oscillator death). For example: if the equations for θ_1 and θ_2 have an attracting critical point. We will see examples where this happens in the problems, e.g.: "Bifurcations in the torus # 01".

10.2 Phase locking and oscillator death

The coupling of two oscillators, each with a stable attracting limit cycle, can produce many behaviors. Two of particular interest are

1. Often, if the frequencies are close enough, the system **phase locks.** This means that a stable periodic solution arises, in which both oscillators run at some composite frequency, with their phase difference kept constant. The composite frequency need not be constant. In fact, it may periodically oscillate about a constant average value.
2. However, the coupling may also suppress the oscillations, with the resulting system having a stable steady state. This even if none of the component oscillators has a stable steady state. This is **oscillator death.** It can happen not only for coupled pairs of oscillators, but also for chains of oscillators with coupling to the nearest neighbors.

On the other hand, we note that it is also possible to produce an oscillating system, with a stable oscillation, by coupling non-oscillating systems (e.g., the coupling of excitable systems can do this).

11 Notes: first order equation with a periodic right hand side

You will be asked to justify the statements below in one of the problems. They are stated here because these results are used/needed in the answers to some of the other problems.

Consider the equation

$$\boldsymbol{\dot{\phi} = 1 - \kappa\,\sin\phi,} \quad \text{where } \boldsymbol{0 < \kappa < 1.} \tag{11.1}$$

Since $\boldsymbol{\dot{\phi} \geq 1 - \kappa > 0}$, ϕ *is monotone increasing.* **It can be shown that:**

1. There is a constant $0 < \mu < 1$, and a function $\Phi = \Phi(\zeta)$ — periodic of period 2π — such that any solution to (11.1) has the form

$$\boldsymbol{\phi = \mu\,(t - t_0) + \Phi(\mu\,(t - t_0))}, \tag{11.2}$$

where $\boldsymbol{t_0}$ is a constant and $\boldsymbol{\Phi(0) = 0}$. It follows that **$\sin(\phi)$ is periodic in t, with period $T = \frac{2\pi}{\mu}$**

2. The period-average M for $\sin(\phi)$ is given by $M = M(\kappa)$ only (μ depends only on κ).

$$\boldsymbol{M = \text{average}(\sin\phi) = \frac{1-\mu}{\kappa} > 0}. \tag{11.3}$$

3. Let **ϕ_* be the solution to (11.1) defined by $\phi_*(0) = 0$** — i.e.: set $t_0 = 0$ in (11.2). Then

$$\Theta(\mu\,t) = \int_0^t (\sin(\phi_*(s)) - M)\,ds = -\frac{1}{\kappa}\,\Phi(\mu\,t), \tag{11.4}$$

where Θ is defined by the first equality.

4. Assume that $0 < \kappa \ll 1$. Then a Poincaré-Lindstedt expansion yields

$$\phi_* = \mu\,t - \kappa\,(1 - \cos(\mu\,t)) + O(\kappa^2) \quad \text{and} \quad \mu = 1 - \frac{1}{2}\,\kappa^2 + O(\kappa^4). \tag{11.5}$$

It follows that $T = 2\pi + \pi\,\kappa^2 + O(\kappa^4)$ and $M = \dfrac{1}{2}\,\kappa + O(\kappa^3)$.

5. Assume that $0 < 1 - \kappa \ll 1$. Then

$$\boldsymbol{\mu = O(\sqrt{1-\kappa})} \tag{11.6}$$

THE END.