Answers to P-Set # 01, 18.385j/2.036j
MIT (Fall 2020)

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1 Get equation from phase line portrait problem #03

1.1 Statement: Get equation from phase line portrait problem #03

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \tag{1.1}$$

where f is some function which is (at least) Lipschitz continuous. Assume that (1.1) has exactly two critical points (i.e.: x_1 and x_2, with $-\infty < x_1 < x_2 < \infty$). Assume also that x_1 is stable and that x_2 is semi-stable.[1] **Is this possible? Does a function $f = f(x)$ yielding this exist?**
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

1.2 Answer: Get equation from phase line portrait problem #03

The answer is yes. Example

$$\boldsymbol{f(x) = -(x+1)\,(x-1)^2}. \tag{1.2}$$

In this case $\boldsymbol{x_1 = -1}$ and $\boldsymbol{x_2 = 1}$.

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2 Get equation from phase line portrait problem #06

2.1 Statement: Get equation from phase line portrait problem #06

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \tag{2.1}$$

where f is some function which is (at least) Lipschitz continuous. Assume that (2.1) has exactly two critical points (i.e.: x_1 and x_2, with $-\infty < x_1 < x_2 < \infty$). Assume also that both critical points are unstable.[2] **Is this possible? Does a function $f = f(x)$ yielding this exist?**
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

2.2 Answer: Get equation from phase line portrait problem #06

The answer is no. Because f is continuous, it must either be: $f(x) > 0$ for $x_1 < x < x_2$ (in which case x_2 cannot be unstable), or $f(x) < 0$ for $x_1 < x < x_2$ (in which case x_1 cannot be unstable).

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[1] A critical point is semi-stable if the solutions diverge from the critical point on one side, and converge on the other.
[2] A critical point is unstable if the solutions diverge from the critical point on both sides.

3 Ill posed initial value problem #01

3.1 Statement: Ill posed initial value problem #01

Consider the initial value problem

$$\frac{dx}{dt} = 2\sqrt{|x|}, \quad \text{for } 0 < t, \quad \text{with } x(0) = 0. \tag{3.1}$$

Show that this problem has infinitely many solutions, which can be parameterized by the time $0 \leq \tau \leq \infty$ beyond which the solution ceases to vanish.

Hint: use separation of variables to find all the solutions that are possible when $x \neq 0$.

3.2 Answer: Ill posed initial value problem #01

1. Separation of variables, assuming that $x > 0$, gives the solutions $\boldsymbol{x = (t - \tau)^2}$, where $t > \tau$ and τ is an arbitrary constant.
2. Separation of variables, assuming that $x < 0$, gives the solutions $\boldsymbol{x = -(t - \tau)^2}$, where $t < \tau$ and τ is an arbitrary constant.

From this we see that (3.1) has the solutions (one solution for each τ)

$$x = 0 \text{ for } 0 \leq t \leq \tau \quad \text{and} \quad x = (t - \tau)^2 \text{ for } \tau \leq t, \tag{3.2}$$

where $0 \leq \tau \leq \infty$ is a constant. Important: this clearly satisfies the ode for $0 \leq t < \tau$, and $\tau < t$. At $t = \tau$ we have to check directly: there both x and $\dot{x}$ are continuous, and satisfy the equation.

Note: because $x(0) = 0$ and $\dot{x} \geq 0$, the solution is always non-negative, and item 2 is not needed.

4 Implicit function problem #01

4.1 Statement: Implicit function problem #01

Consider the following equation

$$\boldsymbol{f(x, \lambda) = x + \lambda \cos(x) = 0}, \tag{4.1}$$

with the particular solution $\boldsymbol{(x, \lambda) = (0, 0)}$. Since $f_x(0, 0) = 1$, the implicit function theorem guarantees that: *there is a neighborhood of $\lambda = 0$ where (4.1) has a unique solution, $x = X(\lambda)$, such that $X(0) = 0$.* Furthermore, since f is an analytic function, X is an analytic function of λ. This means that X has a Taylor series

$$\boldsymbol{X = \sum_{n=0}^{\infty} x_n \lambda^n}, \tag{4.2}$$

which converges for $|\lambda|$ small enough. **Find $\boldsymbol{x_1}$, $\boldsymbol{x_3}$, $\boldsymbol{x_5}$, and $\boldsymbol{x_n}$ for all even $\boldsymbol{n}$.**

4.2 Answer: Implicit function problem #01

First, since $X(0) = 0$, $x_0 = 0$. Note that (4.1) is invariant under the transformation $x \to -x$ and $\lambda \to -\lambda$, which implies that $x = -X(-\lambda)$ is also a solution. From uniqueness it follows that $\boldsymbol{X}$ **is an odd function,** so that

$$\boldsymbol{x_n = 0} \quad \textbf{for all } \boldsymbol{n} \textbf{ even.} \tag{4.3}$$

Next, substitute (4.2) into (4.1), and use the Taylor series for the cosine. Upon collecting equal powers of λ, this yields

$$(1+x_1)\,\lambda + (x_3 - \frac{1}{2}\,x_1^2)\,\lambda^3 + (x_5 - x_1\,x_3 + \frac{1}{24}\,x_1^4)\,\lambda^5 + \cdots = 0. \tag{4.4}$$

Equating to zero the coefficients of the powers of λ, we obtain

$$\boldsymbol{x_1 = -1, \quad x_3 = \frac{1}{2}, \quad x_5 = -\frac{13}{24}, \quad \dots} \tag{4.5}$$

Further coefficients can be obtained by calculating the higher order terms in (4.4). *The implicit function theorem guarantees that the resulting equations will uniquely determine the coefficients x_n.*

5 Inverse function problem #01

5.1 Statement: Inverse function problem #01

Consider the following equation

$$\boldsymbol{y = x + \sin(x) = f(x)}, \tag{5.1}$$

where $\boldsymbol{f(0) = 0}$ and $\boldsymbol{f'(0) = 2 \neq 0}$. The inverse function theorem guarantees that *there is a neighborhood of $x = 0$ where f has a unique inverse, $x = X(y)$, such that $X(0) = 0$.* Furthermore, since f is an analytic function, X is an analytic function. This

means that X has a Taylor series

$$\boldsymbol{X = \sum_{n=0}^{\infty} x_n\, y^n}, \tag{5.2}$$

which converges for $|\lambda|$ small enough. **Find x_1, x_3, x_5, and x_n for all even n.**

5.2 Answer: Inverse function problem #01

Since $X(0) = 0$, $x_0 = 0$. Further, equation (5.1) is invariant under the transformation $x \to -x$ and $y \to -y$, thus $x = -X(-y)$ is also a solution. From uniqueness it follows that **X is an odd function,** so that

$$\boldsymbol{x_n = 0 \quad \text{for all } n \text{ even.}} \tag{5.3}$$

Next, substitute (5.2) into (5.1), and use the Taylor series for the sine. Upon collecting equal powers of y, this yields

$$(2\,x_1 - 1)\,y + (2\,x_3 - \frac{1}{6}\,x_1^3)\,y^3 + (2\,x_5 - \frac{1}{2}\,x_1^2\,x_3 + \frac{1}{120}\,x_1^5)\,y^5 + \cdots = 0. \tag{5.4}$$

Equating to zero the coefficients of the powers of y, we obtain

$$\boldsymbol{x_1 = \frac{1}{2}, \quad x_3 = \frac{1}{96}, \quad x_5 = \frac{1}{1920}, \quad \dots} \tag{5.5}$$

Further coefficients can be obtained by calculating the higher order terms in (5.4). *The implicit function theorem guarantees that the resulting equations will uniquely determine the coefficients x_n.*

6 Linear model for a hanging rope under tension

6.1 Statement: Linear model for a hanging rope under tension

Consider a thin rope stretched (at equilibrium) between two nails at the same height, with the nails separated by some horizontal distance. Idealize the rope as a curve in space, and describe its shape by the vertical deviation of the rope $u = u(x)$ from the horizontal straight line connecting the nails. *Assume that the deviation of the shape from a straight line is small, so that at any point along the rope* $\boldsymbol{\sin(\theta) \approx \tan(\theta) = \frac{du}{dx}}$ *is a good approximation* — here θ is the angle with the horizontal made by the tangent line to the rope.

Let the distance between nails be $\boldsymbol{L}$ (the left nail is at $x = 0$ and the right one at $x = L$), let $\boldsymbol{T > 0}$ be the tension along the rope,[3] and let $\boldsymbol{g}$ be the acceleration of gravity.[4] Finally assume that the rope is homogeneous, with density ρ (mass per unit length).

Write a mathematical model for $\boldsymbol{u}$, and show that it is well posed (the solution is unique, and it depends continuously on all the parameters involved).

Hint #1: The model is a boundary value problem in $0 < x < L$, for a second order, non-homogeneous, linear ode.

Hint #2: To derive the ode for u, consider the balance of the vertical forces[5] *on an arbitrary segment of the rope $a \le x \le b$. The forces are: (i) gravity, (ii) the tension force by the piece of rope on $x > b$, and (iii) the tension force by the piece of rope on $x > a$. Then divide by $(b - a)$ the resulting equation, and take the limit $(b - a) \to 0$ — with both b, $a \to x_0 =$ some arbitrary point along the rope.*

Hint #3: What the boundary conditions on u are is (or should) be obvious.

6.2 Answer: Linear model for a hanging rope under tension

Following *hint #2*, we calculate all the vertical forces on a piece of rope $a < x < b$.

1. The force by gravity is $F_g = -g\,\rho\,(b - a)$, negative because it point downwards.
2. The force by the rope to the right $(x > b)$ is $F_R = \quad T\,\sin(\theta_b) \approx \quad T\,\dfrac{du}{dx}(b)$.
3. The force by the rope to the left[6] $(x < a)$ is $F_L = -T\,\sin(\theta_a) \approx -T\,\dfrac{du}{dx}(a)$.

Balance of the forces $(F_g + F_R + F_L = 0)$ yields

$$T\,\frac{du}{dx}(b) - T\,\frac{du}{dx}(a) = g\,\rho\,(b - a). \tag{6.1}$$

Taking now the limit specified in the hint gives the ode

$$\boldsymbol{\frac{d^2u}{dx^2} = \frac{g\,\rho}{T}} \quad \text{for} \quad \boldsymbol{0 < x < L,} \tag{6.2}$$

which must be satisfy the **boundary conditions $\boldsymbol{u(0) = u(L) = 0}$**. This determines u uniquely:

$$u = -\frac{g\,\rho}{2\,T}\,x\,(L - x). \tag{6.3}$$

In addition, the solution depends continuously on all the involved parameters. Hence the problem is well posed.

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[3] The tension is the magnitude of the force with which, at any point, one side of the rope pulls on the other. The force vector is directed along the rope's tangent, towards the pulling side, and has length T.

[4] Hence, on any mass m, there is a downwards force of magnitude $m\,g$.

[5] Because the rope is at equilibrium, the forces must add to zero.

[6] Note that F_R is positive (upwards) if $\frac{du}{dx}(b) > 0$, while F_L is positive if $\frac{du}{dx}(a) < 0$.

7 Problem 02.02.10 - Strogatz (Fixed points)

7.1 Statement for problem 02.02.10

(Fixed points). For each of (A)–(E) below, find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not. In all cases assume that $f(x)$ is a smooth function.

A. Every real number is a fixed point.

B. Every integer is a fixed point, and there are no others.

C. There are precisely three fixed points, and all of them are stable.

D. There are no fixed points.

E. There are precisely 100 fixed points.

7.2 Answer for problem 02.02.10

We have:

A. Every real number is a fixed point: $\boxed{\dot{x} = 0.}$

B. Every integer is a fixed point, and there are no others: $\boxed{\dot{x} = \sin(\pi x).}$

C. There are precisely three fixed points, and all of them are stable: **Not possible.**
Assume there is an equation $\dot{x} = f(x)$ with these properties, with f smooth. Let the fixed points be $x_0 < x_1 < x_2$. Then, because x_0 is stable, and f has no zeros for $x_0 < x < x_1$, it follows that $f(x) < 0$ for $x_0 < x < x_1$. But the stability of x_1 requires $f(x) > 0$ for $x_0 < x < x_1$. Thus we arrive at a contradiction.

D. There are no fixed points: $\boxed{\dot{x} = 1.}$

E. There are precisely 100 fixed points: $\boxed{\dot{x} = \prod_{n=1}^{100} (x - n).}$

8 Problem 02.05.06 - Strogatz (The leaky bucket)

8.1 Statement for problem 02.05.06

(The leaky bucket). The following example[7] shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see a water bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying[8] a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Here is a crude model for the situation. Let $h(t)$ = height of the water remaining in the bucket at time t; a = area of the hole; A = cross-sectional area of the bucket (assumed constant); $v(t)$ = velocity of the water passing through the hole.

[7] Hubbard, J. H., and West, B. H. (1991) *Differential Equations: A Dynamical Systems Approach, Part I* (Springer, New York).
[8] Note that, in this problem, evaporation effects are neglected.

a. Show that $a\,v(t) = A\,\dot{h}$. What physical law are you invoking? **Warning:** since $\dot{h} < 0$, this presumes that *we assign a negative value to the velocity v*. This is a weird choice, implicit in the problem statement, but acceptable.

b. To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount Δh, and that the water has density ρ. Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation $v^2 = 2\,g\,h$ — g = gravity acceleration.

c. Combining **a** and **b**, show that $\dot{h} = -C\,\sqrt{h}$, where $C = \dfrac{a}{A}\,\sqrt{2\,g}$.

d. Given $h(0) = 0$ (bucket empty at $t = 0$), show that the solution for $h(t)$ is non-unique *in backwards* time, i.e., for $t < 0$.

The description/derivation above ignores surface tension. Briefly discuss the effect surface tension has on the outcome.

8.2 Answer for problem 02.05.06

a. Since water is conserved (*conservation of mass*), the rate at which the water leaves the bucket — i.e.: $a\,v(t)$, must equal the rate at which the water in the bucket decreases — i.e.: $A\,\dot{h}$. Hence: $\boxed{a\,v(t) = A\,\dot{h}.}$ **Warning:** since $\dot{h} < 0$, this presumes that *we assign a negative value to the velocity v*. This is a weird choice, implicit in the problem statement, but acceptable.

b. The potential energy of the water in the bucket is given by $V = \displaystyle\int_0^h g\,A\,\rho\,y\,dy = \frac{1}{2}\,g\,A\,\rho\,h^2$. The rate at which kinetic energy is transported out of the bucket by the escaping water is $\dot{K} = \dfrac{1}{2}\,\dot{m}\,v^2$, where $\dot{m} = \rho\,A\,\dot{h}$ is the rate at which mass leaves the bucket. Hence $\dot{K} = \dfrac{1}{2}\,\rho\,A\,\dot{h}\,v^2$. We now invoke *conservation of energy*, neglecting frictional losses, and equate $\dot{V} = \dot{K} \Longrightarrow \boxed{v^2 = 2\,g\,h.}$

Note that an alternative expression for the rate at which kinetic energy is transported out of the bucket by the escaping water is given by $\dot{K} = \dfrac{1}{2}\,\rho\,a\,v^3$ — since in an infinitesimal time interval dt the kinetic energy transported out is $dK = \dfrac{1}{2}\,(\rho\,a\,v\,dt)\,v^2$. Using **a**, it is easy to see that this is equivalent to the expression derived in the prior paragraph.

c. A simple calculation using **a** and **b** (be careful with the signs, recall that: $v < 0$ and $\dot{h} < 0$) then shows that $\boxed{\dot{h} = -C\,\sqrt{h},}$ where $\boxed{C = \dfrac{a}{A}\,\sqrt{2\,g} > 0,}$ and the *implicit restriction* on the solutions given by $h \geq 0$ applies.

d. Given $h(0) = 0$ (empty bucket), then the solution(s) to the o.d.e. derived in item **c** are:

- Since $\dot{h} \leq 0$, and h must remain non-negative, *the solution is unique for $t > 0$*. Namely: $h(t) \equiv 0$.
- Using separation of variables, we can find the following (infinite number of) solutions, valid for $t < 0$:

$$h(t) = \begin{cases} 0 & \text{for} \quad t_0 \leq t \leq 0, \\ \left(\frac{C}{2}\,(t_0 - t)\right)^2 & \text{for} \quad t \leq t_0. \end{cases}$$

where $t_0 \leq 0$ is arbitrary.

What about surface tension? Surface tension affects this problem in (roughly) three ways: **(i)** On the top surface it will create an extra force that resists the emptying of the bucket. However, by assumption, the top surface is "large", so this is not an important effect [certainly not for a bucket-sized container]. **(ii)** The jet that comes out of the hole may be fractured into droplets by surface tension. Whether or not this happens is not relevant for this problem. Once the water leaves the bucket, it no longer affects h. **(iii)** If the hole at the bottom is small enough, surface tension may be able to stop the flow before $h = 0$. What happens beyond this depends on whether or not the bucket surface is wetting. If it is not, then the interface at the hole will be pinned, with surface tension across it able to support the pressure by a nonzero h in the bucket — a stable, steady, situation. If the bucket surface is wetting, then the interface will spread, allowing more water to flow through the hole, eventually making the interface unstable. Then a drop forms and falls, re-starting the process. The bucket continues to empty out, but not through a jet, but drop-by-drop. Eventually h may be too small to support even this, and this process stops.

9 Problem 02.06.01 - Strogatz (1D oscillator "paradox")

9.1 Statement for problem 02.06.01

Explain this paradox: a simple harmonic oscillator $m\ddot{x} = -kx$ is a system that oscillates in one dimension (along the x-axis). But the text says that one-dimensional systems cannot oscillate.

9.2 Answer for problem 02.06.01

The dimension of a dynamical system (as used in the book) refers to the order of the ODE involved (when the dynamical system is given by an ODE). To be more precise, write the ODE in the cannonical form

$$\frac{d\vec{Y}}{dt} = \vec{F}(\vec{Y}),$$

where $\vec{Y}$ and $\vec{F}$ are vector valued. Then the dimension of the system is the dimension of the vectors $\vec{Y}$ and $\vec{F}$. In particular, for the harmonic oscillator, this yields a dimension of two. Thus, there is no "paradox".

Another way of saying this is that: **the dimension of a dynamical system is given by the dimension of the phase space.**

10 Well posed initial value problem #01

10.1 Statement: Well posed initial value problem #01

Consider the initial value problem

$$\frac{dx}{dt} = -2\,\mathrm{sign}(x)\,\sqrt{|x|}, \quad \text{for } 0 < t, \quad \text{with } x(0) = x_0 = \text{arbitrary constant}. \tag{10.1}$$

Find ALL the solutions to this problem — note that the solutions are defined for **all** times $\boldsymbol{t > 0}$.

Hint: use separation of variables to find all the solutions that are possible when $x \neq 0$.

Notice that the right hand side of the ode above is not Lipschitz continuous at $x = 0$. Nevertheless, this problem is well posed. **Show this, using the solutions that you obtained in the first part of the problem.**

10.2 Answer: Well posed initial value problem #01

1. Separation of variables, assuming $x > 0$, gives $\boldsymbol{x = (t - \tau)^2}$, where $t < \tau =$ arbitrary constant.
2. Separation of variables, assuming $x < 0$, gives $\boldsymbol{x = -(t - \tau)^2}$, where $t < \tau =$ arbitrary constant.

Furthermore, notice that $\dot{x} > 0$ for $x < 0$, and $\dot{x} < 0$ for $x > 0$. Hence, once a solution reaches zero, it must stay there. It follows that (10.1) has the solutions:

$$\boldsymbol{x = \mathrm{sign}(x_0)\,(t - \sqrt{|x_0|})^2} \;\text{ for }\; \boldsymbol{0 < t < \sqrt{|x_0|},} \quad \text{and} \quad \boldsymbol{x = 0} \;\text{ for }\; \boldsymbol{t \geq \sqrt{|x_0|}.} \tag{10.2}$$

This solution is unique, and depends continuously on x_0. Hence *the problem is well posed.* ♣

The presence of the factor $\mathbf{sign}(\boldsymbol{x_0})$ in (10.2) could give the (false) impression that the solution is discontinuous across $x_0 = 0$. However, note that as $x_0 \to 0$ the interval over which the solution does not vanish, $0 < t < \sqrt{|x_0|}$, tends to zero. Furthermore $|x(t)| < |x_0|$. Hence there is no discontinuity — even though the dependence is not differentiable, with respect to x_0, at $x_0 = 0$.

THE END.